Problem 1.32

Find a closed-form solution to the following Riccati equations:

(*e*) $y' + y^2 + (2x+1)y + 1 + x + x^2 = 0.$

Solution

The standard procedure for solving Riccati equations is to guess one solution and then to make an additive substitution to transform it to a Bernoulli equation. Observe that $y_1 = -x$ satisfies the ODE. Hence, we can make the substitution,

$$y(x) = -x + u(x),$$

to transform the ODE to one that is easier to solve. Take a derivative of it to find out what y' is in terms of the new variable.

$$\frac{dy}{dx} = -1 + \frac{du}{dx}$$

Plug these expressions into the Riccati equation.

$$-1 + \frac{du}{dx} + (-x+u)^2 + (2x+1)(-x+u) + 1 + x + x^2 = 0$$

Expand the left side.

$$-1 + \frac{du}{dx} + x^2 + u^2 - 2xu - 2x^2 + 2xu - x + u + 1 + x + x^2 = 0$$

Combine like-terms.

$$\frac{du}{dx} + u^2 + u = 0$$

This is a Bernoulli equation, which has a routine method of solution. Bring u^2 to the right side.

$$\frac{du}{dx} + u = -u^2$$

Divide both sides by u^2.

$$u^{-2}\frac{du}{dx} + u^{-1} = -1$$

Make the substitution,

$$w = u^{-1}$$

Take a derivative of this to find out what u' is in terms of the new variable.

$$\frac{dw}{dx} = -u^{-2}\frac{du}{dx} \quad \rightarrow \quad -\frac{dw}{dx} = u^{-2}\frac{du}{dx}$$

Make these substitutions into the Bernoulli equation.

$$-\frac{dw}{dx} + w = -1$$

This is a first-order inhomogeneous ODE, which can be solved with an integrating factor I. Multiply both sides by -1 first to get it into standard form.

$$\frac{dw}{dx} - w = 1$$

The integrating factor is this.

$$I = e^{\int^x (-1)\,ds} = e^{-x}$$

Multiply both sides of the equation by I.

$$e^{-x}\frac{dw}{dx} - e^{-x}w = e^{-x}$$

The ODE is now exact, and the left side can be written as $d/dx(Iw)$ as a result of the product rule.

$$\frac{d}{dx}(e^{-x}w) = e^{-x}$$

Integrate both sides with respect to x.

$$e^{-x}w = -e^{-x} + C$$

Multiply both sides by e^x.

$$w(x) = -1 + Ce^x$$

Now that w is solved for, change back to u.

$$\frac{1}{u} = -1 + Ce^x$$

Invert both sides.

$$u(x) = \frac{1}{-1 + Ce^x}$$

Now that u is solved for, change back to the original variable y.

$$y + x = \frac{1}{-1 + Ce^x}$$

Subtract x from both sides to solve for y. Therefore, we have

$$y(x) = -x + \frac{1}{-1 + Ce^x}$$

for the general solution to the Riccati equation.