Problem 1.33

Under what conditions does the differential equation $y' = f(x, y)$ have an integrating factor of the form $I(xy)$?

Solution

Bring $f(x, y)$ to the left side.

$$-f(x,y) + \frac{dy}{dx} = 0$$

Multiply both sides by the integrating factor $I(xy)$.

$$-I(xy)f(x,y) + I(xy)\frac{dy}{dx} = 0$$

In order for the ODE to be exact, we require that

$$\frac{\partial}{\partial y}[-I(xy)f(x,y)] = \frac{\partial}{\partial x}[I(xy)].$$

Evaluate the partial derivative on each side.

$$-xI'(xy)f(x,y) - I(xy)\frac{\partial f}{\partial y} = yI'(xy)$$

Bring $xI'f$ to the right side and factor I'.

$$-I(xy)\frac{\partial f}{\partial y} = [y + xf(x,y)]I'(xy)$$

Make the substitution,

$$s = xy \quad \rightarrow \quad \frac{s}{x} = y.$$

We have to write f_y in terms of the new variable now using the chain rule.

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial s}\frac{\partial s}{\partial y} = x\frac{\partial f}{\partial s}$$

Plugging these expressions into the ODE, it becomes

$$-I(s)\left(x\frac{\partial f}{\partial s}\right) = \left(\frac{s}{x} + xf\right)\frac{dI}{ds}.$$

Multiply both sides by x.

$$-I(s)\left(x^2\frac{\partial f}{\partial s}\right) = \left(s + x^2 f\right)\frac{dI}{ds} \tag{1}$$

In order for this to be a legitimate ODE for I, only s can be present in the equation. We need $x^2 f_s$ to be some function of s and $x^2 f$ to be a function of s as well. That is,

$$x^2 f(x,s) = F(s),$$

where F is an arbitrary function of s. Changing back to the original variable y, we see that the function f has to have the form,

$$f(x,y) = \frac{F(xy)}{x^2},$$

in order for $I(xy)$ to be an appropriate integrating factor. One can solve (1) to find I given f.

$$-I(s)F'(s) = [s + F(s)]\frac{dI}{ds}$$