Problem 3.12

Show that

$$\frac{\frac{x}{1}+\frac{x^3}{1\cdot 3}+\frac{x^5}{1\cdot 3\cdot 5}+\frac{x^7}{1\cdot 3\cdot 5\cdot 7}+\cdots}{1+\frac{x^2}{2}+\frac{x^4}{2\cdot 4}+\frac{x^6}{2\cdot 4\cdot 6}+\cdots}=\int_0^x e^{-t^2/2}\,dt$$

(Putnam Exam 1950).

Solution

We will start with the quotient of series on the left side and derive the integral on the right side. The first step is to represent the infinite series compactly in sums. To make it easier to see how, rewrite the quotient as follows.

$$\frac{\frac{x}{1}+\frac{2}{1\cdot 2\cdot 3}x^3+\frac{2\cdot 4}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^5+\frac{2\cdot 4\cdot 6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}x^7+\cdots}{1+\frac{x^2}{2}+\frac{x^4}{2\cdot 4}+\frac{x^6}{2\cdot 4\cdot 6}+\cdots}$$

To write a product of numbers that skips every other term, we use a double factorial.

$$\frac{\sum_{n=0}^{\infty}\frac{(2n)!!}{(2n+1)!}x^{2n+1}}{\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!!}}$$

Since $2n$ is an even number, we can write $(2n)!!$ in terms of the regular factorial as $2^n n!$.

$$\frac{\sum_{n=0}^{\infty}\frac{2^n n!}{(2n+1)!}x^{2n+1}}{\sum_{n=0}^{\infty}\frac{x^{2n}}{2^n n!}}$$

The series in the denominator is actually the Maclaurin series for the exponential function.

$$\sum_{n=0}^{\infty}\frac{x^{2n}}{2^n n!}=\sum_{n=0}^{\infty}\frac{1}{n!}\cdot\left(\frac{x^2}{2}\right)^n=e^{x^2/2}$$

Therefore,

$$\frac{\sum_{n=0}^{\infty}\frac{2^n n!}{(2n+1)!}x^{2n+1}}{e^{x^2/2}}.$$

Bring the exponential function to the numerator by making its exponent negative.

$$e^{-x^2/2}\sum_{n=0}^{\infty}\frac{2^n n!}{(2n+1)!}x^{2n+1}$$

Now take the derivative of this expression with respect to x. The motivation for doing this comes from the fact that there's an integral on the right side of the equation we have to prove. If we differentiate both sides, the integral gets eliminated and we're left with $e^{-x^2/2}$ on the right. That's what we aim to end up with by differentiating the expression we have now. Make use of the product rule here—the first function is the exponential and the second function is the series.

$$-xe^{-x^2/2}\sum_{n=0}^{\infty}\frac{2^n n!}{(2n+1)!}x^{2n+1}+e^{-x^2/2}\sum_{n=0}^{\infty}\frac{2^n n!}{(2n)!}x^{2n}$$

Factor out the exponential function and bring the x term into the summand.

$$e^{-x^2/2}\left[-\sum_{n=0}^{\infty}\frac{2^n n!}{(2n+1)!}x^{2n+2}+\sum_{n=0}^{\infty}\frac{2^n n!}{(2n)!}x^{2n}\right]$$

Since both series start from $n=0$, they can be combined into one.

$$e^{-x^2/2}\sum_{n=0}^{\infty}\left[\frac{2^n n!}{(2n)!}x^{2n}-\frac{2^n n!}{(2n+1)!}x^{2n+2}\right]$$

In order to evaluate the sum, write out the first three terms of the series.

$$e^{-x^2/2}\left(\underbrace{1-\cancel{x^2}}_{n=0}+\underbrace{\cancel{x^2}-\cancel{\frac{x^4}{3}}}_{n=1}+\underbrace{\cancel{\frac{x^4}{3}}-\cancel{\frac{x^6}{15}}}_{n=2}+\underbrace{\cancel{\frac{x^6}{15}}-\frac{x^8}{105}}_{n=3}+\cdots\right)$$

Each value of n gives us two terms. The second term of n always cancels with the first term of $n+1$. Hence, this is a telescoping series whose sum we can determine by calculating a limit.

$$e^{-x^2/2}\left(1-\lim_{n\to\infty}\frac{2^n n!}{(2n+1)!}x^{2n+2}\right)$$

Since the denominator grows and the numerator stays at 1 as n gets big, the limit tends to 0. The expression thus simplifies to

$$e^{-x^2/2}.$$

We have shown so far that

$$\frac{d}{dx}\left(\frac{\frac{x}{1}+\frac{x^3}{1\cdot 3}+\frac{x^5}{1\cdot 3\cdot 5}+\frac{x^7}{1\cdot 3\cdot 5\cdot 7}+\cdots}{1+\frac{x^2}{2}+\frac{x^4}{2\cdot 4}+\frac{x^6}{2\cdot 4\cdot 6}+\cdots}\right)=e^{-x^2/2}.$$

Integrate both sides now with respect to x to obtain the desired result.

$$\frac{\frac{x}{1}+\frac{x^3}{1\cdot 3}+\frac{x^5}{1\cdot 3\cdot 5}+\frac{x^7}{1\cdot 3\cdot 5\cdot 7}+\cdots}{1+\frac{x^2}{2}+\frac{x^4}{2\cdot 4}+\frac{x^6}{2\cdot 4\cdot 6}+\cdots}=\int_0^x e^{-t^2/2}\,dt$$

The lower limit of integration is set to 0 because when we plug in $x=0$, we have to have the same value (zero) on both sides of the equation.