Exercise 3

If $\boldsymbol{\alpha}$ is symmetrical and $\boldsymbol{\beta}$ is antisymmetrical, show that $(\boldsymbol{\alpha} : \boldsymbol{\beta}) = 0$.

Solution

$\boldsymbol{\alpha}$ is symmetrical implies that $\alpha_{ij} = \alpha_{ji}$, and $\boldsymbol{\beta}$ is antisymmetrical implies that $\beta_{ij} = -\beta_{ji}$.

$$\begin{aligned}
\boldsymbol{\alpha} : \boldsymbol{\beta} = \left(\sum_{i=1}^{3}\sum_{j=1}^{3} \boldsymbol{\delta}_i\boldsymbol{\delta}_j\alpha_{ij}\right) : \left(\sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_k\boldsymbol{\delta}_l\beta_{kl}\right) &= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} (\boldsymbol{\delta}_i\boldsymbol{\delta}_j : \boldsymbol{\delta}_k\boldsymbol{\delta}_l)\alpha_{ij}\beta_{kl} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_l)(\boldsymbol{\delta}_j \cdot \boldsymbol{\delta}_k)\alpha_{ij}\beta_{kl} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \delta_{il}\delta_{jk}\alpha_{ij}\beta_{kl} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \delta_{jk}\alpha_{ij}\beta_{ki} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} \alpha_{ij}\beta_{ji} \\
&= \sum_{i=1}^{3} (\alpha_{i1}\beta_{1i} + \alpha_{i2}\beta_{2i} + \alpha_{i3}\beta_{3i}) \\
&= \alpha_{11}\beta_{11} + \alpha_{12}\beta_{21} + \alpha_{13}\beta_{31} \\
&\quad + \alpha_{21}\beta_{12} + \alpha_{22}\beta_{22} + \alpha_{23}\beta_{32} \\
&\quad + \alpha_{31}\beta_{13} + \alpha_{32}\beta_{23} + \alpha_{33}\beta_{33}
\end{aligned}$$

Switch the indices of α in the lower left.

$$\begin{aligned}
&= \alpha_{11}\beta_{11} + \alpha_{12}\beta_{21} + \alpha_{13}\beta_{31} \\
&\quad + \alpha_{12}\beta_{12} + \alpha_{22}\beta_{22} + \alpha_{23}\beta_{32} \\
&\quad + \alpha_{13}\beta_{13} + \alpha_{23}\beta_{23} + \alpha_{33}\beta_{33}
\end{aligned}$$

Switch the indices of β in the lower left. Many terms cancel as a result.

$$\begin{aligned}
&= \alpha_{11}\beta_{11} + \cancel{\alpha_{12}\beta_{21}} + \cancel{\alpha_{13}\beta_{31}} \\
&\quad - \cancel{\alpha_{12}\beta_{21}} + \alpha_{22}\beta_{22} + \cancel{\alpha_{23}\beta_{32}} \\
&\quad - \cancel{\alpha_{13}\beta_{31}} - \cancel{\alpha_{23}\beta_{32}} + \alpha_{33}\beta_{33}
\end{aligned}$$

Switch the indices of β.

$$\begin{aligned}
&= -\alpha_{11}\beta_{11} - \alpha_{22}\beta_{22} - \alpha_{33}\beta_{33} \\
&= -(\alpha_{11}\beta_{11} + \alpha_{22}\beta_{22} + \alpha_{33}\beta_{33})
\end{aligned}$$

The only number equal to its negative is zero.

$$= 0$$