Exercise 4

A vector $\mathbf{v}$ has components

$$v_i = \sum_{j=1}^{3} \alpha_{ij} x_j$$

with $\alpha_{ij} = \alpha_{ji}$ and $\sum_{i=1}^{3} \alpha_{ii} = 0$; the α_{ij} are constants. Evaluate $(\nabla \cdot \mathbf{v})$, $[\nabla \times \mathbf{v}]$, $\nabla \mathbf{v}$, $(\nabla \mathbf{v})^\dagger$, and $[\nabla \cdot \mathbf{v}\mathbf{v}]$. (*Hint*: In connection with evaluating $[\nabla \times \mathbf{v}]$, see Exercise 5 in §A.2.)

Solution

Evaluate the divergence of $\mathbf{v}$.

$$\begin{aligned}\nabla \cdot \mathbf{v} &= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j v_j\right) = \sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j) \frac{\partial v_j}{\partial x_i} = \sum_{i=1}^{3}\sum_{j=1}^{3} \delta_{ij} \frac{\partial v_j}{\partial x_i} = \sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i} \\ &= \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{3} \alpha_{ij} x_j\right) = \sum_{i=1}^{3}\sum_{j=1}^{3} \alpha_{ij} \frac{\partial x_j}{\partial x_i} = \sum_{i=1}^{3}\sum_{j=1}^{3} \alpha_{ij} \delta_{ij} = \sum_{i=1}^{3} \alpha_{ii} = 0\end{aligned}$$

Evaluate the curl of $\mathbf{v}$.

$$\begin{aligned}\nabla \times \mathbf{v} &= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j v_j\right) = \sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \times \boldsymbol{\delta}_j) \frac{\partial v_j}{\partial x_i} = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{ijk} \frac{\partial v_j}{\partial x_i} \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{ijk} \frac{\partial}{\partial x_i}\left(\sum_{l=1}^{3} \alpha_{jl} x_l\right) = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_k \varepsilon_{ijk} \alpha_{jl} \frac{\partial x_l}{\partial x_i} \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_k \varepsilon_{ijk} \alpha_{jl} \delta_{li} = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{ijk} \alpha_{ji} = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{ijk} \alpha_{ij} \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{kij} \alpha_{ij} = \sum_{k=1}^{3} \boldsymbol{\delta}_k \left(\sum_{i=1}^{3}\sum_{j=1}^{3} \varepsilon_{kij} \alpha_{ij}\right) = \sum_{k=1}^{3} \boldsymbol{\delta}_k (0) = \mathbf{0},\end{aligned}$$

where the double sum in parentheses is 0 as a result of Exercise 5 in §A.2. Evaluate the gradient of $\mathbf{v}$ now.

$$\begin{aligned}\nabla \mathbf{v} &= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right)\left(\sum_{j=1}^{3} \boldsymbol{\delta}_j v_j\right) = \sum_{i=1}^{3}\sum_{j=1}^{3} \boldsymbol{\delta}_i \boldsymbol{\delta}_j \frac{\partial v_j}{\partial x_i} = \sum_{i=1}^{3}\sum_{j=1}^{3} \boldsymbol{\delta}_i \boldsymbol{\delta}_j \frac{\partial}{\partial x_i}\left(\sum_{k=1}^{3} \alpha_{jk} x_k\right) \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_i \boldsymbol{\delta}_j \alpha_{jk} \frac{\partial x_k}{\partial x_i} = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_i \boldsymbol{\delta}_j \alpha_{jk} \delta_{ki} = \sum_{i=1}^{3}\sum_{j=1}^{3} \boldsymbol{\delta}_i \boldsymbol{\delta}_j \alpha_{ji} = \sum_{i=1}^{3}\sum_{j=1}^{3} \boldsymbol{\delta}_i \boldsymbol{\delta}_j \alpha_{ij} = \boldsymbol{\alpha},\end{aligned}$$

where $\boldsymbol{\alpha}$ is a second-order tensor with α_{ij} for the components. Since $\alpha_{ij} = \alpha_{ji}$, this tensor is symmetric. Taking the transpose of $\nabla \mathbf{v}$ therefore does not change the result.

$$(\nabla \mathbf{v})^\dagger = (\boldsymbol{\alpha})^\dagger = \boldsymbol{\alpha}$$

Evaluate the divergence of the dyadic product **vv**.

$$
\begin{aligned}
\nabla\cdot\mathbf{vv} &= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\cdot\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j v_j\right)\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k v_k\right) = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}(\boldsymbol{\delta}_i\cdot\boldsymbol{\delta}_j)\boldsymbol{\delta}_k\frac{\partial}{\partial x_i}(v_j v_k)\\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\delta_{ij}\boldsymbol{\delta}_k\frac{\partial}{\partial x_i}(v_j v_k) = \sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_k\frac{\partial}{\partial x_j}(v_j v_k)\\
&= \sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_k\frac{\partial}{\partial x_j}\left[\left(\sum_{l=1}^{3}\alpha_{jl}x_l\right)\left(\sum_{m=1}^{3}\alpha_{km}x_m\right)\right] = \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_k\alpha_{jl}\alpha_{km}\frac{\partial}{\partial x_j}(x_l x_m)\\
&= \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_k\alpha_{jl}\alpha_{km}\left(\frac{\partial x_l}{\partial x_j}x_m + x_l\frac{\partial x_m}{\partial x_j}\right) = \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_k\alpha_{jl}\alpha_{km}(\delta_{lj}x_m + x_l\delta_{mj})\\
&= \sum_{k=1}^{3}\boldsymbol{\delta}_k\left[\sum_{j=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\alpha_{jl}\alpha_{km}(\delta_{lj}x_m + x_l\delta_{mj})\right]\\
&= \sum_{k=1}^{3}\boldsymbol{\delta}_k\left(\sum_{j=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\alpha_{jl}\alpha_{km}\delta_{lj}x_m + \sum_{j=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\alpha_{jl}\alpha_{km}x_l\delta_{mj}\right)\\
&= \sum_{k=1}^{3}\boldsymbol{\delta}_k\left(\sum_{j=1}^{3}\sum_{m=1}^{3}\alpha_{jj}\alpha_{km}x_m + \sum_{j=1}^{3}\sum_{l=1}^{3}\alpha_{jl}\alpha_{kj}x_l\right)\\
&= \sum_{k=1}^{3}\boldsymbol{\delta}_k\left[\left(\sum_{j=1}^{3}\alpha_{jj}\right)\sum_{m=1}^{3}\alpha_{km}x_m + \sum_{j=1}^{3}\sum_{l=1}^{3}\alpha_{jl}\alpha_{kj}x_l\right]\\
&= \sum_{k=1}^{3}\boldsymbol{\delta}_k\left[(0)\sum_{m=1}^{3}\alpha_{km}x_m + \sum_{j=1}^{3}\sum_{l=1}^{3}\alpha_{jl}\alpha_{kj}x_l\right]\\
&= \sum_{k=1}^{3}\boldsymbol{\delta}_k\left(\sum_{j=1}^{3}\sum_{l=1}^{3}\alpha_{jl}\alpha_{kj}x_l\right)
\end{aligned}
$$

We thus have a vector whose k-component is given by the double sum in parentheses.