Exercise 5

Verify that $\nabla^2(\nabla \cdot \mathbf{v}) = (\nabla \cdot (\nabla^2 \mathbf{v}))$, and that $[\nabla \cdot (\nabla \mathbf{v})^\dagger] = \nabla(\nabla \cdot \mathbf{v})$.

Solution

The First Vector Identity

$$
\begin{aligned}
\nabla^2(\nabla \cdot \mathbf{v}) = (\nabla \cdot \nabla)(\nabla \cdot \mathbf{v}) &= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial}{\partial x_j}\right) \left(\sum_{k=1}^{3} \boldsymbol{\delta}_k \frac{\partial}{\partial x_k}\right) \cdot \left(\sum_{l=1}^{3} \boldsymbol{\delta}_l v_l\right) \\
&= \left[\sum_{i=1}^{3} \sum_{j=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}\right] \left[\sum_{k=1}^{3} \sum_{l=1}^{3} (\boldsymbol{\delta}_k \cdot \boldsymbol{\delta}_l) \frac{\partial v_l}{\partial x_k}\right] \\
&= \left(\sum_{i=1}^{3} \sum_{j=1}^{3} \delta_{ij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}\right) \left(\sum_{k=1}^{3} \sum_{l=1}^{3} \delta_{kl} \frac{\partial v_l}{\partial x_k}\right) = \left(\sum_{i=1}^{3} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i}\right) \left(\sum_{k=1}^{3} \frac{\partial v_k}{\partial x_k}\right) \\
&= \sum_{i=1}^{3} \sum_{k=1}^{3} \frac{\partial^2}{\partial x_i^2} \left(\frac{\partial v_k}{\partial x_k}\right) = \sum_{i=1}^{3} \sum_{k=1}^{3} \frac{\partial}{\partial x_k} \left(\frac{\partial^2 v_k}{\partial x_i^2}\right) = \sum_{k=1}^{3} \frac{\partial}{\partial x_k} \left(\sum_{i=1}^{3} \frac{\partial^2 v_k}{\partial x_i^2}\right) \\
&= \sum_{k=1}^{3} \frac{\partial}{\partial x_k} (\nabla^2 v_k) = \nabla \cdot (\nabla^2 \mathbf{v})
\end{aligned}
$$

The Second Vector Identity

$$
\begin{aligned}
\nabla \cdot (\nabla \mathbf{v})^\dagger &= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left[\left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial}{\partial x_j}\right) \left(\sum_{k=1}^{3} \boldsymbol{\delta}_k v_k\right)\right]^\dagger = \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left(\sum_{j=1}^{3} \sum_{k=1}^{3} \boldsymbol{\delta}_j \boldsymbol{\delta}_k \frac{\partial v_k}{\partial x_j}\right)^\dagger \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left(\sum_{j=1}^{3} \sum_{k=1}^{3} \boldsymbol{\delta}_j \boldsymbol{\delta}_k \frac{\partial v_j}{\partial x_k}\right) = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_j) \boldsymbol{\delta}_k \frac{\partial}{\partial x_i} \left(\frac{\partial v_j}{\partial x_k}\right) \\
&= \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \delta_{ij} \boldsymbol{\delta}_k \frac{\partial}{\partial x_i} \left(\frac{\partial v_j}{\partial x_k}\right) = \sum_{j=1}^{3} \sum_{k=1}^{3} \boldsymbol{\delta}_k \frac{\partial}{\partial x_j} \left(\frac{\partial v_j}{\partial x_k}\right) = \sum_{j=1}^{3} \sum_{k=1}^{3} \boldsymbol{\delta}_k \frac{\partial}{\partial x_k} \left(\frac{\partial v_j}{\partial x_j}\right) \\
&= \sum_{k=1}^{3} \boldsymbol{\delta}_k \frac{\partial}{\partial x_k} \left(\sum_{j=1}^{3} \frac{\partial v_j}{\partial x_j}\right) = \sum_{k=1}^{3} \boldsymbol{\delta}_k \frac{\partial}{\partial x_k} (\nabla \cdot \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v})
\end{aligned}
$$

Both vector identities have been successfully verified.