Exercise 4

Use Eq. A.5-4 (with $\mathbf{v}$ replaced by $\boldsymbol{\tau}$) to show that, when $\tau_{ki} = \sum_j \varepsilon_{ijk} x_j$,

$$2\int_S \mathbf{n}\, dS = \oint_C [\mathbf{r} \times \mathbf{t}]\, dC$$

where $\mathbf{r}$ is the position vector locating a point on C with respect to the origin.

Solution

Replacing $\mathbf{v}$ with $\boldsymbol{\tau}$ in Eq. A.5-4 (Stokes's theorem) gives us

$$\iint_S (\mathbf{n} \cdot [\nabla \times \boldsymbol{\tau}])\, dS = \oint_C (\mathbf{t} \cdot \boldsymbol{\tau})\, dC,$$

where $\mathbf{t}$ is a unit vector tangent to the integration path C and $\mathbf{n}$ is a unit vector in the direction the thumb points when the fingers of the right hand curl in the direction of the path. The fact that $\mathbf{r}$ is the position vector means its components are x_i.

$$\mathbf{r} = \sum_{i=1}^{3} \boldsymbol{\delta}_i x_i$$

<u>The Left-hand Side</u>

$$\begin{aligned}
\iint_S (\mathbf{n} \cdot [\nabla \times \boldsymbol{\tau}])\, dS &= \iint_S \left(\sum_{l=1}^{3} \boldsymbol{\delta}_l n_l\right) \cdot \left[\left(\sum_{m=1}^{3} \boldsymbol{\delta}_m \frac{\partial}{\partial x_m}\right) \times \left(\sum_{k=1}^{3}\sum_{i=1}^{3} \boldsymbol{\delta}_k \boldsymbol{\delta}_i \tau_{ki}\right)\right] dS \\
&= \iint_S \left(\sum_{l=1}^{3} \boldsymbol{\delta}_l n_l\right) \cdot \left[\sum_{m=1}^{3}\sum_{k=1}^{3}\sum_{i=1}^{3} (\boldsymbol{\delta}_m \times \boldsymbol{\delta}_k) \boldsymbol{\delta}_i \frac{\partial}{\partial x_m}(\tau_{ki})\right] dS \\
&= \iint_S \left(\sum_{l=1}^{3} \boldsymbol{\delta}_l n_l\right) \cdot \left[\sum_{m=1}^{3}\sum_{k=1}^{3}\sum_{i=1}^{3}\sum_{p=1}^{3} \boldsymbol{\delta}_p \varepsilon_{mkp} \boldsymbol{\delta}_i \frac{\partial}{\partial x_m}(\tau_{ki})\right] dS \\
&= \iint_S \left[\sum_{l=1}^{3}\sum_{m=1}^{3}\sum_{k=1}^{3}\sum_{i=1}^{3}\sum_{p=1}^{3} (\boldsymbol{\delta}_l \cdot \boldsymbol{\delta}_p) \varepsilon_{mkp} \boldsymbol{\delta}_i n_l \frac{\partial}{\partial x_m}(\tau_{ki})\right] dS \\
&= \iint_S \left[\sum_{l=1}^{3}\sum_{m=1}^{3}\sum_{k=1}^{3}\sum_{i=1}^{3}\sum_{p=1}^{3} \delta_{lp} \varepsilon_{mkp} \boldsymbol{\delta}_i n_l \frac{\partial}{\partial x_m}(\tau_{ki})\right] dS \\
&= \iint_S \left[\sum_{m=1}^{3}\sum_{k=1}^{3}\sum_{i=1}^{3}\sum_{p=1}^{3} \varepsilon_{mkp} \boldsymbol{\delta}_i n_p \frac{\partial}{\partial x_m}(\tau_{ki})\right] dS \\
&= \iint_S \left[\sum_{m=1}^{3}\sum_{k=1}^{3}\sum_{i=1}^{3}\sum_{p=1}^{3} \varepsilon_{mkp} \boldsymbol{\delta}_i n_p \frac{\partial}{\partial x_m}\left(\sum_{j=1}^{3} \varepsilon_{ijk} x_j\right)\right] dS \\
&= \iint_S \left(\sum_{m=1}^{3}\sum_{k=1}^{3}\sum_{i=1}^{3}\sum_{p=1}^{3}\sum_{j=1}^{3} \varepsilon_{ijk} \varepsilon_{mkp} \boldsymbol{\delta}_i n_p \frac{\partial x_j}{\partial x_m}\right) dS
\end{aligned}$$

$$\begin{aligned}
\iint_S (\mathbf{n} \cdot [\nabla \times \boldsymbol{\tau}])\, dS &= \iint_S \left(\sum_{m=1}^{3} \sum_{k=1}^{3} \sum_{i=1}^{3} \sum_{p=1}^{3} \sum_{j=1}^{3} \varepsilon_{ijk}\varepsilon_{pmk} \boldsymbol{\delta}_i n_p \delta_{jm} \right) dS \\
&= \iint_S \left(\sum_{k=1}^{3} \sum_{i=1}^{3} \sum_{p=1}^{3} \sum_{j=1}^{3} \varepsilon_{ijk}\varepsilon_{pjk} \boldsymbol{\delta}_i n_p \right) dS \\
&= \iint_S \left[\sum_{i=1}^{3} \sum_{p=1}^{3} (2\delta_{ip}) \boldsymbol{\delta}_i n_p \right] dS \\
&= \iint_S \left(\sum_{i=1}^{3} 2\boldsymbol{\delta}_i n_i \right) dS \\
&= \iint_S 2\mathbf{n}\, dS \\
&= 2 \iint_S \mathbf{n}\, dS
\end{aligned}$$

The Right-hand Side

$$\begin{aligned}
\oint_C [\mathbf{t} \cdot \boldsymbol{\tau}]\, dC &= \oint_C \left[\left(\sum_{l=1}^{3} \boldsymbol{\delta}_l t_l \right) \cdot \left(\sum_{k=1}^{3} \sum_{i=1}^{3} \boldsymbol{\delta}_k \boldsymbol{\delta}_i \tau_{ki} \right) \right] dC \\
&= \oint_C \left[\sum_{l=1}^{3} \sum_{k=1}^{3} \sum_{i=1}^{3} (\boldsymbol{\delta}_l \cdot \boldsymbol{\delta}_k) \boldsymbol{\delta}_i t_l \tau_{ki} \right] dC \\
&= \oint_C \left(\sum_{l=1}^{3} \sum_{k=1}^{3} \sum_{i=1}^{3} \delta_{lk} \boldsymbol{\delta}_i t_l \tau_{ki} \right) dC \\
&= \oint_C \left(\sum_{k=1}^{3} \sum_{i=1}^{3} \boldsymbol{\delta}_i t_k \tau_{ki} \right) dC \\
&= \oint_C \left[\sum_{k=1}^{3} \sum_{i=1}^{3} \boldsymbol{\delta}_i t_k \left(\sum_{j=1}^{3} \varepsilon_{ijk} x_j \right) \right] dC \\
&= \oint_C \left(\sum_{k=1}^{3} \sum_{i=1}^{3} \sum_{j=1}^{3} \boldsymbol{\delta}_i \varepsilon_{ijk} x_j t_k \right) dC \\
&= \oint_C \left(\sum_{k=1}^{3} \sum_{i=1}^{3} \sum_{j=1}^{3} \boldsymbol{\delta}_i \varepsilon_{jki} x_j t_k \right) dC \\
&= \oint_C (\mathbf{r} \times \mathbf{t})\, dC
\end{aligned}$$

Therefore, by Stokes's theorem,

$$2 \iint_S \mathbf{n}\, dS = \oint_C (\mathbf{r} \times \mathbf{t})\, dC.$$