Exercise 5

Evaluate both sides of Eq. A.5-2 for the function $s(x, y, z) = x^2 + y^2 + z^2$. The volume V is the triangular prism lying between the two triangles whose vertices are $(2, 0, 0), (2, 1, 0), (2, 0, 3)$, and $(-2, 0, 0), (-2, 1, 0), (-2, 0, 3)$.

Solution

Eq. A.5-2 is the divergence theorem for scalars (as opposed to vectors),

$$\iiint_V \nabla s \, dV = \oiint_S \mathbf{n} s \, dS,$$

where V is a closed volume and $\mathbf{n}$ is a unit vector normal to its surface in the outward direction.

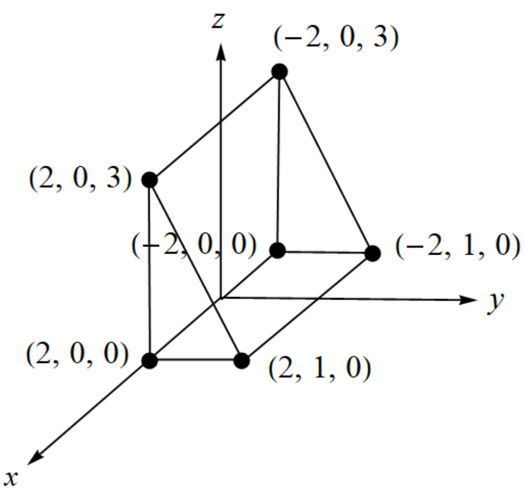

Figure 1: Schematic of the triangular prism with the given vertices.

The Left-hand Side

$$\begin{aligned}
\iiint_V \nabla s \, dV &= \iiint_V \nabla(x^2 + y^2 + z^2) \, dV \\
&= \iiint_V (2x\boldsymbol{\delta}_x + 2y\boldsymbol{\delta}_y + 2z\boldsymbol{\delta}_z) \, dV \\
&= \int_{-2}^{2} \int_0^1 \int_0^{3(1-y)} (2x\boldsymbol{\delta}_x + 2y\boldsymbol{\delta}_y + 2z\boldsymbol{\delta}_z) \, dz \, dy \, dx \\
&= \int_{-2}^{2} \int_0^1 (2xz\boldsymbol{\delta}_x + 2yz\boldsymbol{\delta}_y + z^2\boldsymbol{\delta}_z)\Big|_0^{3(1-y)} \, dy \, dx \\
&= \int_{-2}^{2} \int_0^1 [6x(1-y)\boldsymbol{\delta}_x + 6y(1-y)\boldsymbol{\delta}_y + 9(1-y)^2\boldsymbol{\delta}_z] \, dy \, dx \\
&= \int_{-2}^{2} \left[6x\left(y - \frac{y^2}{2}\right)\boldsymbol{\delta}_x + 6\left(\frac{y^2}{2} - \frac{y^3}{3}\right)\boldsymbol{\delta}_y + 9\left(y - y^2 + \frac{y^3}{3}\right)\boldsymbol{\delta}_z\right]\Bigg|_0^1 \, dx \\
&= \int_{-2}^{2} (3x\boldsymbol{\delta}_x + \boldsymbol{\delta}_y + 3\boldsymbol{\delta}_z) \, dx
\end{aligned}$$

$$\iiint_V \nabla s\, dV = \left(3\frac{x^2}{2}\boldsymbol{\delta}_x + x\boldsymbol{\delta}_y + 3x\boldsymbol{\delta}_z\right)\Bigg|_{-2}^{2}$$
$$= 0\boldsymbol{\delta}_x + 4\boldsymbol{\delta}_y + 12\boldsymbol{\delta}_z$$

The Right-hand Side

The triangular prism has five faces, so the closed surface integral will split up into five double integrals.

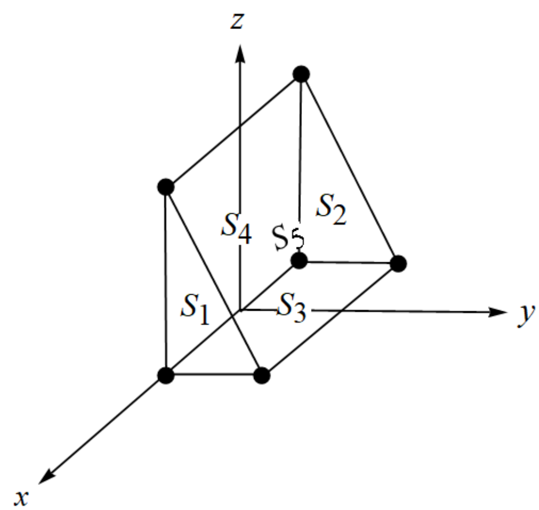

Figure 2: Schematic of the triangular prism with labeled faces.

$$\oiint_S \mathbf{n}s\, dS = \iint_{S_1} \mathbf{n}s\, dS + \iint_{S_2} \mathbf{n}s\, dS + \iint_{S_3} \mathbf{n}s\, dS + \iint_{S_4} \mathbf{n}s\, dS + \iint_{S_5} \mathbf{n}s\, dS$$

The outward unit vector normal to S_1 is $\boldsymbol{\delta}_x$, the outward unit vector normal to S_2 is $-\boldsymbol{\delta}_x$, the outward unit vector normal to S_3 is $-\boldsymbol{\delta}_z$, the outward unit vector normal to S_4 is $-\boldsymbol{\delta}_y$, and the outward unit vector normal to S_5 is $0\boldsymbol{\delta}_x + 3\boldsymbol{\delta}_y + \boldsymbol{\delta}_z$ divided by its magnitude.

$$\oiint_S \mathbf{n}s\, dS = \iint_{S_1} \boldsymbol{\delta}_x s\, dS + \iint_{S_2} (-\boldsymbol{\delta}_x)s\, dS + \iint_{S_3} (-\boldsymbol{\delta}_z)s\, dS + \iint_{S_4} (-\boldsymbol{\delta}_y)s\, dS + \iint_{S_5} \frac{3\boldsymbol{\delta}_y + \boldsymbol{\delta}_z}{\sqrt{3^2+1^2}} s\, dS$$

The double integrals over S_1 and S_2 will be in dy and dz, the double integrals over S_3 and S_5 will be in dx and dy, and the double integral over S_4 will be in dx and dz.

$$\begin{aligned}\oiint_S \mathbf{n}s\, dS = \int_0^1 \int_0^{3(1-y)} \boldsymbol{\delta}_x s\, dz\, dy + \int_0^1 \int_0^{3(1-y)} (-\boldsymbol{\delta}_x)s\, dz\, dy + \int_{-2}^2 \int_0^1 (-\boldsymbol{\delta}_z)s\, dy\, dx \\ + \int_{-2}^2 \int_0^3 (-\boldsymbol{\delta}_y)s\, dz\, dx + \int_{-2}^2 \int_0^1 \frac{3\boldsymbol{\delta}_y + \boldsymbol{\delta}_z}{\sqrt{3^2+1^2}} s(\sqrt{3^2+1^2})\, dy\, dx\end{aligned}$$

Factor out the unit vectors and bring the constants in front.

$$\oiint_S \mathbf{n}s\,dS = \boldsymbol{\delta}_x \left[\underbrace{\int_0^1 \int_0^{3(1-y)} s\,dz\,dy}_{S_1} - \underbrace{\int_0^1 \int_0^{3(1-y)} s\,dz\,dy}_{S_2}\right] + \boldsymbol{\delta}_y \left[-\underbrace{\int_{-2}^2 \int_0^3 s\,dz\,dx}_{S_4} + 3\underbrace{\int_{-2}^2 \int_0^1 s\,dy\,dx}_{S_5}\right] + \boldsymbol{\delta}_z \left[-\underbrace{\int_{-2}^2 \int_0^1 s\,dy\,dx}_{S_3} + \underbrace{\int_{-2}^2 \int_0^1 s\,dy\,dx}_{S_5}\right]$$

On S_1, $x = 2$; on S_2, $x = -2$; on S_3, $z = 0$; on S_4, $y = 0$; and on S_5, $z = 3 - 3y$.

$$\oiint_S \mathbf{n}s\,dS = \boldsymbol{\delta}_x \left[\int_0^1 \int_0^{3(1-y)} (2^2 + y^2 + z^2)\,dz\,dy - \int_0^1 \int_0^{3(1-y)} [(-2)^2 + y^2 + z^2]\,dz\,dy\right] + \boldsymbol{\delta}_y \left[-\int_{-2}^2 \int_0^3 (x^2 + 0^2 + z^2)\,dz\,dx + 3\int_{-2}^2 \int_0^1 [x^2 + y^2 + (3 - 3y)^2]\,dy\,dx\right] + \boldsymbol{\delta}_z \left[-\int_{-2}^2 \int_0^1 (x^2 + y^2 + 0^2)\,dy\,dx + \int_{-2}^2 \int_0^1 [x^2 + y^2 + (3 - 3y)^2]\,dy\,dx\right]$$

The integrands of the first two double integrals are the same, so the integrals cancel.

$$\oiint_S \mathbf{n}s\,dS = 0\boldsymbol{\delta}_x + \boldsymbol{\delta}_y \left[-\int_{-2}^2 \left(x^2 z + \frac{z^3}{3}\right)\Big|_0^3 dx + 3\int_{-2}^2 \left[x^2 y + \frac{y^3}{3} + (9y - 9y^2 + 3y^3)\right]\Big|_0^1 dx\right] + \boldsymbol{\delta}_z \left[-\int_{-2}^2 \left(x^2 y + \frac{y^3}{3}\right)\Big|_0^1 dx + \int_{-2}^2 \left[x^2 y + \frac{y^3}{3} + (9y - 9y^2 + 3y^3)\right]\Big|_0^1 dx\right]$$

Plug in the limits and simplify.

$$\oiint_S \mathbf{n}s\,dS = 0\boldsymbol{\delta}_x + \boldsymbol{\delta}_y \left[-\int_{-2}^2 (3x^2 + 9)\,dx + 3\int_{-2}^2 \left(x^2 + \frac{10}{3}\right) dx\right] + \boldsymbol{\delta}_z \left[-\int_{-2}^2 \left(x^2 + \frac{1}{3}\right) dx + \int_{-2}^2 \left(x^2 + \frac{10}{3}\right) dx\right]$$

Evaluate the single integrals.

$$\oiint_S \mathbf{n}s\,dS = 0\boldsymbol{\delta}_x + 4\boldsymbol{\delta}_y + 12\boldsymbol{\delta}_z$$

We conclude that the divergence theorem for scalars is verified.