Exercise 2

Obtain $(\nabla \cdot \mathbf{v})$, $[\nabla \times \mathbf{v}]$, and $\nabla \mathbf{v}$ in spherical coordinates, and $[\nabla \cdot \boldsymbol{\tau}]$ in cylindrical coordinates.

Solution

In spherical coordinates

$$\nabla = \boldsymbol{\delta}_r \frac{\partial}{\partial r} + \boldsymbol{\delta}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{\delta}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

and

$$\mathbf{v} = \boldsymbol{\delta}_r v_r + \boldsymbol{\delta}_\theta v_\theta + \boldsymbol{\delta}_\phi v_\phi.$$

The partial derivatives of $\boldsymbol{\delta}_r$, $\boldsymbol{\delta}_\theta$, and $\boldsymbol{\delta}_\phi$ in spherical coordinates are given by equations A.7-6, A.7-7, and A.7-8,

$$\frac{\partial \boldsymbol{\delta}_r}{\partial r} = 0 \qquad \frac{\partial \boldsymbol{\delta}_\theta}{\partial r} = 0 \qquad \frac{\partial \boldsymbol{\delta}_\phi}{\partial r} = 0 \tag{A.7-6}$$

$$\frac{\partial \boldsymbol{\delta}_r}{\partial \theta} = \boldsymbol{\delta}_\theta \qquad \frac{\partial \boldsymbol{\delta}_\theta}{\partial \theta} = -\boldsymbol{\delta}_r \qquad \frac{\partial \boldsymbol{\delta}_\phi}{\partial \theta} = 0 \tag{A.7-7}$$

$$\frac{\partial \boldsymbol{\delta}_r}{\partial \phi} = \boldsymbol{\delta}_\phi \sin\theta \qquad \frac{\partial \boldsymbol{\delta}_\theta}{\partial \phi} = \boldsymbol{\delta}_\phi \cos\theta \qquad \frac{\partial \boldsymbol{\delta}_\phi}{\partial \phi} = -\boldsymbol{\delta}_r \sin\theta - \boldsymbol{\delta}_\theta \cos\theta. \tag{A.7-8}$$

The Divergence of v

$$\begin{aligned}
\nabla \cdot \mathbf{v} &= \left(\boldsymbol{\delta}_r \frac{\partial}{\partial r} + \boldsymbol{\delta}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{\delta}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right) \cdot (\boldsymbol{\delta}_r v_r + \boldsymbol{\delta}_\theta v_\theta + \boldsymbol{\delta}_\phi v_\phi) \\
&= \boldsymbol{\delta}_r \cdot \frac{\partial}{\partial r}(\boldsymbol{\delta}_r v_r + \boldsymbol{\delta}_\theta v_\theta + \boldsymbol{\delta}_\phi v_\phi) \\
&\qquad + \boldsymbol{\delta}_\theta \cdot \frac{1}{r}\frac{\partial}{\partial \theta}(\boldsymbol{\delta}_r v_r + \boldsymbol{\delta}_\theta v_\theta + \boldsymbol{\delta}_\phi v_\phi) \\
&\qquad + \boldsymbol{\delta}_\phi \cdot \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(\boldsymbol{\delta}_r v_r + \boldsymbol{\delta}_\theta v_\theta + \boldsymbol{\delta}_\phi v_\phi) \\
&= \boldsymbol{\delta}_r \cdot \left(\frac{\partial \boldsymbol{\delta}_r}{\partial r} v_r + \boldsymbol{\delta}_r \frac{\partial v_r}{\partial r} + \frac{\partial \boldsymbol{\delta}_\theta}{\partial r} v_\theta + \boldsymbol{\delta}_\theta \frac{\partial v_\theta}{\partial r} + \frac{\partial \boldsymbol{\delta}_\phi}{\partial r} v_\phi + \boldsymbol{\delta}_\phi \frac{\partial v_\phi}{\partial r}\right) \\
&\qquad + \boldsymbol{\delta}_\theta \cdot \frac{1}{r}\left(\frac{\partial \boldsymbol{\delta}_r}{\partial \theta} v_r + \boldsymbol{\delta}_r \frac{\partial v_r}{\partial \theta} + \frac{\partial \boldsymbol{\delta}_\theta}{\partial \theta} v_\theta + \boldsymbol{\delta}_\theta \frac{\partial v_\theta}{\partial \theta} + \frac{\partial \boldsymbol{\delta}_\phi}{\partial \theta} v_\phi + \boldsymbol{\delta}_\phi \frac{\partial v_\phi}{\partial \theta}\right) \\
&\qquad + \boldsymbol{\delta}_\phi \cdot \frac{1}{r\sin\theta}\left(\frac{\partial \boldsymbol{\delta}_r}{\partial \phi} v_r + \boldsymbol{\delta}_r \frac{\partial v_r}{\partial \phi} + \frac{\partial \boldsymbol{\delta}_\theta}{\partial \phi} v_\theta + \boldsymbol{\delta}_\theta \frac{\partial v_\theta}{\partial \phi} + \frac{\partial \boldsymbol{\delta}_\phi}{\partial \phi} v_\phi + \boldsymbol{\delta}_\phi \frac{\partial v_\phi}{\partial \phi}\right)
\end{aligned}$$

Using equations A.7-6, A.7-7, and A.7-8, we get

$$\begin{aligned}
\nabla \cdot \mathbf{v} = \boldsymbol{\delta}_r &\cdot \left(\boldsymbol{\delta}_r \frac{\partial v_r}{\partial r} + \boldsymbol{\delta}_\theta \frac{\partial v_\theta}{\partial r} + \boldsymbol{\delta}_\phi \frac{\partial v_\phi}{\partial r}\right) \\
&+ \boldsymbol{\delta}_\theta \cdot \frac{1}{r}\left(\boldsymbol{\delta}_\theta v_r + \boldsymbol{\delta}_r \frac{\partial v_r}{\partial \theta} - \boldsymbol{\delta}_r v_\theta + \boldsymbol{\delta}_\theta \frac{\partial v_\theta}{\partial \theta} + \boldsymbol{\delta}_\phi \frac{\partial v_\phi}{\partial \theta}\right) \\
&+ \boldsymbol{\delta}_\phi \cdot \frac{1}{r\sin\theta}\left[\boldsymbol{\delta}_\phi \sin\theta v_r + \boldsymbol{\delta}_r \frac{\partial v_r}{\partial \phi} + \boldsymbol{\delta}_\phi \cos\theta v_\theta + \boldsymbol{\delta}_\theta \frac{\partial v_\theta}{\partial \phi} + (-\boldsymbol{\delta}_r \sin\theta - \boldsymbol{\delta}_\theta \cos\theta) v_\phi + \boldsymbol{\delta}_\phi \frac{\partial v_\phi}{\partial \phi}\right].
\end{aligned}$$

Evaluate the three dot products.

$$\nabla \cdot \mathbf{v} = \frac{\partial v_r}{\partial r} + \frac{1}{r}\left(v_r + \frac{\partial v_\theta}{\partial \theta}\right) + \frac{1}{r\sin\theta}\left(\sin\theta v_r + \cos\theta v_\theta + \frac{\partial v_\phi}{\partial \phi}\right)$$

Expand the terms.

$$\nabla \cdot \mathbf{v} = \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} + \frac{\cos\theta}{r\sin\theta}v_\theta + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Combine like-terms and factor.

$$\nabla \cdot \mathbf{v} = \frac{\partial v_r}{\partial r} + \frac{2}{r}v_r + \frac{1}{r\sin\theta}\left(\sin\theta\frac{\partial v_\theta}{\partial \theta} + \cos\theta v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Factor $1/r^2$ from the first two terms.

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\left(r^2\frac{\partial v_r}{\partial r} + 2rv_r\right) + \frac{1}{r\sin\theta}\left(\sin\theta\frac{\partial v_\theta}{\partial \theta} + \cos\theta v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Using the product rule, we obtain the final result for the divergence of $\mathbf{v}$ in spherical coordinates.

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(v_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

The Curl of v

$$\begin{aligned}
\nabla \times \mathbf{v} &= \left(\boldsymbol{\delta}_r\frac{\partial}{\partial r} + \boldsymbol{\delta}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{\delta}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right) \times (\boldsymbol{\delta}_r v_r + \boldsymbol{\delta}_\theta v_\theta + \boldsymbol{\delta}_\phi v_\phi) \\
&= \boldsymbol{\delta}_r \times \frac{\partial}{\partial r}(\boldsymbol{\delta}_r v_r + \boldsymbol{\delta}_\theta v_\theta + \boldsymbol{\delta}_\phi v_\phi) \\
&\quad + \boldsymbol{\delta}_\theta \times \frac{1}{r}\frac{\partial}{\partial \theta}(\boldsymbol{\delta}_r v_r + \boldsymbol{\delta}_\theta v_\theta + \boldsymbol{\delta}_\phi v_\phi) \\
&\quad + \boldsymbol{\delta}_\phi \times \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(\boldsymbol{\delta}_r v_r + \boldsymbol{\delta}_\theta v_\theta + \boldsymbol{\delta}_\phi v_\phi) \\
&= \boldsymbol{\delta}_r \times \left(\frac{\partial \boldsymbol{\delta}_r}{\partial r}v_r + \boldsymbol{\delta}_r\frac{\partial v_r}{\partial r} + \frac{\partial \boldsymbol{\delta}_\theta}{\partial r}v_\theta + \boldsymbol{\delta}_\theta\frac{\partial v_\theta}{\partial r} + \frac{\partial \boldsymbol{\delta}_\phi}{\partial r}v_\phi + \boldsymbol{\delta}_\phi\frac{\partial v_\phi}{\partial r}\right) \\
&\quad + \boldsymbol{\delta}_\theta \times \frac{1}{r}\left(\frac{\partial \boldsymbol{\delta}_r}{\partial \theta}v_r + \boldsymbol{\delta}_r\frac{\partial v_r}{\partial \theta} + \frac{\partial \boldsymbol{\delta}_\theta}{\partial \theta}v_\theta + \boldsymbol{\delta}_\theta\frac{\partial v_\theta}{\partial \theta} + \frac{\partial \boldsymbol{\delta}_\phi}{\partial \theta}v_\phi + \boldsymbol{\delta}_\phi\frac{\partial v_\phi}{\partial \theta}\right) \\
&\quad + \boldsymbol{\delta}_\phi \times \frac{1}{r\sin\theta}\left(\frac{\partial \boldsymbol{\delta}_r}{\partial \phi}v_r + \boldsymbol{\delta}_r\frac{\partial v_r}{\partial \phi} + \frac{\partial \boldsymbol{\delta}_\theta}{\partial \phi}v_\theta + \boldsymbol{\delta}_\theta\frac{\partial v_\theta}{\partial \phi} + \frac{\partial \boldsymbol{\delta}_\phi}{\partial \phi}v_\phi + \boldsymbol{\delta}_\phi\frac{\partial v_\phi}{\partial \phi}\right)
\end{aligned}$$

Using equations A.7-6, A.7-7, and A.7-8, we get

$$\begin{aligned}
\nabla \times \mathbf{v} = \boldsymbol{\delta}_r \times \left(\boldsymbol{\delta}_r\frac{\partial v_r}{\partial r} + \boldsymbol{\delta}_\theta\frac{\partial v_\theta}{\partial r} + \boldsymbol{\delta}_\phi\frac{\partial v_\phi}{\partial r}\right) \\
+ \boldsymbol{\delta}_\theta \times \frac{1}{r}\left(\boldsymbol{\delta}_\theta v_r + \boldsymbol{\delta}_r\frac{\partial v_r}{\partial \theta} - \boldsymbol{\delta}_r v_\theta + \boldsymbol{\delta}_\theta\frac{\partial v_\theta}{\partial \theta} + \boldsymbol{\delta}_\phi\frac{\partial v_\phi}{\partial \theta}\right) \\
+ \boldsymbol{\delta}_\phi \times \frac{1}{r\sin\theta}\left[\boldsymbol{\delta}_\phi\sin\theta v_r + \boldsymbol{\delta}_r\frac{\partial v_r}{\partial \phi} + \boldsymbol{\delta}_\phi\cos\theta v_\theta + \boldsymbol{\delta}_\theta\frac{\partial v_\theta}{\partial \phi} + (-\boldsymbol{\delta}_r\sin\theta - \boldsymbol{\delta}_\theta\cos\theta)v_\phi + \boldsymbol{\delta}_\phi\frac{\partial v_\phi}{\partial \phi}\right].
\end{aligned}$$

Evaluate the cross products.

$$\begin{aligned}\nabla \times \mathbf{v} = \boldsymbol{\delta}_\phi \frac{\partial v_\theta}{\partial r} - \boldsymbol{\delta}_\theta \frac{\partial v_\phi}{\partial r} \\ + \frac{1}{r}\left(-\boldsymbol{\delta}_\phi \frac{\partial v_r}{\partial \theta} + \boldsymbol{\delta}_\phi v_\theta + \boldsymbol{\delta}_r \frac{\partial v_\phi}{\partial \theta}\right) \\ + \frac{1}{r\sin\theta}\left(\boldsymbol{\delta}_\theta \frac{\partial v_r}{\partial \phi} - \boldsymbol{\delta}_r \frac{\partial v_\theta}{\partial \phi} - \boldsymbol{\delta}_\theta \sin\theta v_\phi + \boldsymbol{\delta}_r \cos\theta v_\phi\right)\end{aligned}$$

Combine the like-terms.

$$\begin{aligned}\nabla \times \mathbf{v} = \boldsymbol{\delta}_r \left(\frac{1}{r}\frac{\partial v_\phi}{\partial \theta} - \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{\cos\theta}{r\sin\theta} v_\phi\right) \\ + \boldsymbol{\delta}_\theta \left(-\frac{\partial v_\phi}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{1}{r} v_\phi\right) \\ + \boldsymbol{\delta}_\phi \left(\frac{\partial v_\theta}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{1}{r} v_\theta\right)\end{aligned}$$

Factor the expressions.

$$\begin{aligned}\nabla \times \mathbf{v} = \boldsymbol{\delta}_r \left[\frac{1}{r\sin\theta}\left(\sin\theta \frac{\partial v_\phi}{\partial \theta} + \cos\theta v_\phi\right) - \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right] \\ + \boldsymbol{\delta}_\theta \left[-\frac{1}{r}\left(r\frac{\partial v_\phi}{\partial r} + v_\phi\right) + \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi}\right] \\ + \boldsymbol{\delta}_\phi \left[\frac{1}{r}\left(r\frac{\partial v_\theta}{\partial r} + v_\theta\right) - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]\end{aligned}$$

Using the product rule, we obtain the final result for the curl of $\mathbf{v}$ in spherical coordinates.

$$\begin{aligned}\nabla \times \mathbf{v} = \frac{\boldsymbol{\delta}_r}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(v_\phi \sin\theta) - \frac{\partial v_\theta}{\partial \phi}\right] \\ + \frac{\boldsymbol{\delta}_\theta}{r}\left[-\frac{\partial}{\partial r}(r v_\phi) + \frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi}\right] \\ + \frac{\boldsymbol{\delta}_\phi}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\end{aligned}$$

The Gradient of v

$$\begin{aligned}\nabla\mathbf{v} &= \left(\boldsymbol{\delta}_r\frac{\partial}{\partial r} + \boldsymbol{\delta}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \boldsymbol{\delta}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)(\boldsymbol{\delta}_r v_r + \boldsymbol{\delta}_\theta v_\theta + \boldsymbol{\delta}_\phi v_\phi) \\ &= \boldsymbol{\delta}_r\frac{\partial}{\partial r}(\boldsymbol{\delta}_r v_r + \boldsymbol{\delta}_\theta v_\theta + \boldsymbol{\delta}_\phi v_\phi) \\ &\quad + \boldsymbol{\delta}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}(\boldsymbol{\delta}_r v_r + \boldsymbol{\delta}_\theta v_\theta + \boldsymbol{\delta}_\phi v_\phi) \\ &\quad + \boldsymbol{\delta}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\boldsymbol{\delta}_r v_r + \boldsymbol{\delta}_\theta v_\theta + \boldsymbol{\delta}_\phi v_\phi)\end{aligned}$$

Apply the product rule.

$$\begin{aligned}&= \boldsymbol{\delta}_r\left(\frac{\partial\boldsymbol{\delta}_r}{\partial r}v_r + \boldsymbol{\delta}_r\frac{\partial v_r}{\partial r} + \frac{\partial\boldsymbol{\delta}_\theta}{\partial r}v_\theta + \boldsymbol{\delta}_\theta\frac{\partial v_\theta}{\partial r} + \frac{\partial\boldsymbol{\delta}_\phi}{\partial r}v_\phi + \boldsymbol{\delta}_\phi\frac{\partial v_\phi}{\partial r}\right) \\ &\quad + \frac{\boldsymbol{\delta}_\theta}{r}\left(\frac{\partial\boldsymbol{\delta}_r}{\partial\theta}v_r + \boldsymbol{\delta}_r\frac{\partial v_r}{\partial\theta} + \frac{\partial\boldsymbol{\delta}_\theta}{\partial\theta}v_\theta + \boldsymbol{\delta}_\theta\frac{\partial v_\theta}{\partial\theta} + \frac{\partial\boldsymbol{\delta}_\phi}{\partial\theta}v_\phi + \boldsymbol{\delta}_\phi\frac{\partial v_\phi}{\partial\theta}\right) \\ &\quad + \frac{\boldsymbol{\delta}_\phi}{r\sin\theta}\left(\frac{\partial\boldsymbol{\delta}_r}{\partial\phi}v_r + \boldsymbol{\delta}_r\frac{\partial v_r}{\partial\phi} + \frac{\partial\boldsymbol{\delta}_\theta}{\partial\phi}v_\theta + \boldsymbol{\delta}_\theta\frac{\partial v_\theta}{\partial\phi} + \frac{\partial\boldsymbol{\delta}_\phi}{\partial\phi}v_\phi + \boldsymbol{\delta}_\phi\frac{\partial v_\phi}{\partial\phi}\right)\end{aligned}$$

Using equations A.7-6, A.7-7, and A.7-8, we get

$$\begin{aligned}\nabla\mathbf{v} &= \boldsymbol{\delta}_r\left(\boldsymbol{\delta}_r\frac{\partial v_r}{\partial r} + \boldsymbol{\delta}_\theta\frac{\partial v_\theta}{\partial r} + \boldsymbol{\delta}_\phi\frac{\partial v_\phi}{\partial r}\right) \\ &\quad + \frac{\boldsymbol{\delta}_\theta}{r}\left(\boldsymbol{\delta}_\theta v_r + \boldsymbol{\delta}_r\frac{\partial v_r}{\partial\theta} - \boldsymbol{\delta}_r v_\theta + \boldsymbol{\delta}_\theta\frac{\partial v_\theta}{\partial\theta} + \boldsymbol{\delta}_\phi\frac{\partial v_\phi}{\partial\theta}\right) \\ &\quad + \frac{\boldsymbol{\delta}_\phi}{r\sin\theta}\left[\boldsymbol{\delta}_\phi\sin\theta v_r + \boldsymbol{\delta}_r\frac{\partial v_r}{\partial\phi} + \boldsymbol{\delta}_\phi\cos\theta v_\theta + \boldsymbol{\delta}_\theta\frac{\partial v_\theta}{\partial\phi} + (-\boldsymbol{\delta}_r\sin\theta - \boldsymbol{\delta}_\theta\cos\theta)v_\phi + \boldsymbol{\delta}_\phi\frac{\partial v_\phi}{\partial\phi}\right].\end{aligned}$$

Distribute the unit vectors.

$$\begin{aligned}\nabla\mathbf{v} &= \boldsymbol{\delta}_r\boldsymbol{\delta}_r\frac{\partial v_r}{\partial r} + \boldsymbol{\delta}_r\boldsymbol{\delta}_\theta\frac{\partial v_\theta}{\partial r} + \boldsymbol{\delta}_r\boldsymbol{\delta}_\phi\frac{\partial v_\phi}{\partial r} \\ &\quad + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_r\left(\frac{1}{r}\frac{\partial v_r}{\partial\theta} - \frac{1}{r}v_\theta\right) + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta\left(\frac{1}{r}v_r + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta}\right) + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\phi\frac{1}{r}\frac{\partial v_\phi}{\partial\theta} \\ &\quad + \boldsymbol{\delta}_\phi\boldsymbol{\delta}_r\left(\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{1}{r}v_\phi\right) + \boldsymbol{\delta}_\phi\boldsymbol{\delta}_\theta\left(\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi} - \frac{\cos\theta}{r\sin\theta}v_\phi\right) + \boldsymbol{\delta}_\phi\boldsymbol{\delta}_\phi\left(\frac{1}{r}v_r + \frac{\cos\theta}{r\sin\theta}v_\theta + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right)\end{aligned}$$

Factoring, we obtain the final result for the gradient of $\mathbf{v}$ in spherical coordinates.

$$\begin{aligned}\nabla\mathbf{v} &= \boldsymbol{\delta}_r\boldsymbol{\delta}_r\frac{\partial v_r}{\partial r} + \boldsymbol{\delta}_r\boldsymbol{\delta}_\theta\frac{\partial v_\theta}{\partial r} + \boldsymbol{\delta}_r\boldsymbol{\delta}_\phi\frac{\partial v_\phi}{\partial r} \\ &\quad + \frac{\boldsymbol{\delta}_\theta\boldsymbol{\delta}_r}{r}\left(\frac{\partial v_r}{\partial\theta} - v_\theta\right) + \frac{\boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta}{r}\left(v_r + \frac{\partial v_\theta}{\partial\theta}\right) + \frac{\boldsymbol{\delta}_\theta\boldsymbol{\delta}_\phi}{r}\frac{\partial v_\phi}{\partial\theta} \\ &\quad + \frac{\boldsymbol{\delta}_\phi\boldsymbol{\delta}_r}{r}\left(\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - v_\phi\right) + \frac{\boldsymbol{\delta}_\phi\boldsymbol{\delta}_\theta}{r\sin\theta}\left(\frac{\partial v_\theta}{\partial\phi} - v_\phi\cos\theta\right) + \frac{\boldsymbol{\delta}_\phi\boldsymbol{\delta}_\phi}{r}\left[v_r + \frac{1}{\sin\theta}\left(v_\theta\cos\theta + \frac{\partial v_\phi}{\partial\phi}\right)\right]\end{aligned}$$

The Divergence of $\boldsymbol{\tau}$

In cylindrical coordinates

$$\nabla = \boldsymbol{\delta}_r \frac{\partial}{\partial r} + \boldsymbol{\delta}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \boldsymbol{\delta}_z \frac{\partial}{\partial z}$$

and

$$\begin{aligned}\boldsymbol{\tau} = {} & \boldsymbol{\delta}_r\boldsymbol{\delta}_r\tau_{rr} + \boldsymbol{\delta}_r\boldsymbol{\delta}_\theta\tau_{r\theta} + \boldsymbol{\delta}_r\boldsymbol{\delta}_z\tau_{rz} \\ & + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_r\tau_{\theta r} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta\tau_{\theta\theta} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_z\tau_{\theta z} \\ & + \boldsymbol{\delta}_z\boldsymbol{\delta}_r\tau_{zr} + \boldsymbol{\delta}_z\boldsymbol{\delta}_\theta\tau_{z\theta} + \boldsymbol{\delta}_z\boldsymbol{\delta}_z\tau_{zz}.\end{aligned}$$

The partial derivatives of $\boldsymbol{\delta}_r$, $\boldsymbol{\delta}_\theta$, and $\boldsymbol{\delta}_z$ in cylindrical coordinates are given by equations A.7-1, A.7-2, and A.7-3,

$$\frac{\partial \boldsymbol{\delta}_r}{\partial r} = 0 \qquad \frac{\partial \boldsymbol{\delta}_\theta}{\partial r} = 0 \qquad \frac{\partial \boldsymbol{\delta}_z}{\partial r} = 0 \tag{A.7-1}$$

$$\frac{\partial \boldsymbol{\delta}_r}{\partial \theta} = \boldsymbol{\delta}_\theta \qquad \frac{\partial \boldsymbol{\delta}_\theta}{\partial \theta} = -\boldsymbol{\delta}_r \qquad \frac{\partial \boldsymbol{\delta}_z}{\partial \theta} = 0 \tag{A.7-2}$$

$$\frac{\partial \boldsymbol{\delta}_r}{\partial z} = 0 \qquad \frac{\partial \boldsymbol{\delta}_\theta}{\partial z} = 0 \qquad \frac{\partial \boldsymbol{\delta}_z}{\partial z} = 0. \tag{A.7-3}$$

$$\begin{aligned}\nabla \cdot \boldsymbol{\tau} = {} & \left(\boldsymbol{\delta}_r \frac{\partial}{\partial r} + \boldsymbol{\delta}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \boldsymbol{\delta}_z \frac{\partial}{\partial z}\right) \cdot (\boldsymbol{\delta}_r\boldsymbol{\delta}_r\tau_{rr} + \boldsymbol{\delta}_r\boldsymbol{\delta}_\theta\tau_{r\theta} + \boldsymbol{\delta}_r\boldsymbol{\delta}_z\tau_{rz} \\ & + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_r\tau_{\theta r} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta\tau_{\theta\theta} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_z\tau_{\theta z} \\ & + \boldsymbol{\delta}_z\boldsymbol{\delta}_r\tau_{zr} + \boldsymbol{\delta}_z\boldsymbol{\delta}_\theta\tau_{z\theta} + \boldsymbol{\delta}_z\boldsymbol{\delta}_z\tau_{zz}) \\ = {} & \boldsymbol{\delta}_r \cdot \frac{\partial}{\partial r}(\boldsymbol{\delta}_r\boldsymbol{\delta}_r\tau_{rr} + \boldsymbol{\delta}_r\boldsymbol{\delta}_\theta\tau_{r\theta} + \boldsymbol{\delta}_r\boldsymbol{\delta}_z\tau_{rz} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_r\tau_{\theta r} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta\tau_{\theta\theta} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_z\tau_{\theta z} + \boldsymbol{\delta}_z\boldsymbol{\delta}_r\tau_{zr} + \boldsymbol{\delta}_z\boldsymbol{\delta}_\theta\tau_{z\theta} + \boldsymbol{\delta}_z\boldsymbol{\delta}_z\tau_{zz}) \\ & + \frac{\boldsymbol{\delta}_\theta}{r} \cdot \frac{\partial}{\partial \theta}(\boldsymbol{\delta}_r\boldsymbol{\delta}_r\tau_{rr} + \boldsymbol{\delta}_r\boldsymbol{\delta}_\theta\tau_{r\theta} + \boldsymbol{\delta}_r\boldsymbol{\delta}_z\tau_{rz} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_r\tau_{\theta r} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta\tau_{\theta\theta} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_z\tau_{\theta z} + \boldsymbol{\delta}_z\boldsymbol{\delta}_r\tau_{zr} + \boldsymbol{\delta}_z\boldsymbol{\delta}_\theta\tau_{z\theta} + \boldsymbol{\delta}_z\boldsymbol{\delta}_z\tau_{zz}) \\ & + \boldsymbol{\delta}_z \cdot \frac{\partial}{\partial z}(\boldsymbol{\delta}_r\boldsymbol{\delta}_r\tau_{rr} + \boldsymbol{\delta}_r\boldsymbol{\delta}_\theta\tau_{r\theta} + \boldsymbol{\delta}_r\boldsymbol{\delta}_z\tau_{rz} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_r\tau_{\theta r} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta\tau_{\theta\theta} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_z\tau_{\theta z} + \boldsymbol{\delta}_z\boldsymbol{\delta}_r\tau_{zr} + \boldsymbol{\delta}_z\boldsymbol{\delta}_\theta\tau_{z\theta} + \boldsymbol{\delta}_z\boldsymbol{\delta}_z\tau_{zz})\end{aligned}$$

Apply the product rule.

$$
\begin{aligned}
\nabla\cdot\boldsymbol{\tau} = \boldsymbol{\delta}_r\cdot\Bigg(&\frac{\partial\boldsymbol{\delta}_r}{\partial r}\boldsymbol{\delta}_r\tau_{rr} + \boldsymbol{\delta}_r\frac{\partial\boldsymbol{\delta}_r}{\partial r}\tau_{rr} + \boldsymbol{\delta}_r\boldsymbol{\delta}_r\frac{\partial\tau_{rr}}{\partial r} + \frac{\partial\boldsymbol{\delta}_r}{\partial r}\boldsymbol{\delta}_\theta\tau_{r\theta} + \boldsymbol{\delta}_r\frac{\partial\boldsymbol{\delta}_\theta}{\partial r}\tau_{r\theta} + \boldsymbol{\delta}_r\boldsymbol{\delta}_\theta\frac{\partial\tau_{r\theta}}{\partial r}\\
&+ \frac{\partial\boldsymbol{\delta}_r}{\partial r}\boldsymbol{\delta}_z\tau_{rz} + \boldsymbol{\delta}_r\frac{\partial\boldsymbol{\delta}_z}{\partial r}\tau_{rz} + \boldsymbol{\delta}_r\boldsymbol{\delta}_z\frac{\partial\tau_{rz}}{\partial r} + \frac{\partial\boldsymbol{\delta}_\theta}{\partial r}\boldsymbol{\delta}_r\tau_{\theta r} + \boldsymbol{\delta}_\theta\frac{\partial\boldsymbol{\delta}_r}{\partial r}\tau_{\theta r} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_r\frac{\partial\tau_{\theta r}}{\partial r}\\
&+ \frac{\partial\boldsymbol{\delta}_\theta}{\partial r}\boldsymbol{\delta}_\theta\tau_{\theta\theta} + \boldsymbol{\delta}_\theta\frac{\partial\boldsymbol{\delta}_\theta}{\partial r}\tau_{\theta\theta} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta\frac{\partial\tau_{\theta\theta}}{\partial r} + \frac{\partial\boldsymbol{\delta}_\theta}{\partial r}\boldsymbol{\delta}_z\tau_{\theta z} + \boldsymbol{\delta}_\theta\frac{\partial\boldsymbol{\delta}_z}{\partial r}\tau_{\theta z} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_z\frac{\partial\tau_{\theta z}}{\partial r}\\
&+ \frac{\partial\boldsymbol{\delta}_z}{\partial r}\boldsymbol{\delta}_r\tau_{zr} + \boldsymbol{\delta}_z\frac{\partial\boldsymbol{\delta}_r}{\partial r}\tau_{zr} + \boldsymbol{\delta}_z\boldsymbol{\delta}_r\frac{\partial\tau_{zr}}{\partial r} + \frac{\partial\boldsymbol{\delta}_z}{\partial r}\boldsymbol{\delta}_\theta\tau_{z\theta} + \boldsymbol{\delta}_z\frac{\partial\boldsymbol{\delta}_\theta}{\partial r}\tau_{z\theta} + \boldsymbol{\delta}_z\boldsymbol{\delta}_\theta\frac{\partial\tau_{z\theta}}{\partial r}\\
&+ \frac{\partial\boldsymbol{\delta}_z}{\partial r}\boldsymbol{\delta}_z\tau_{zz} + \boldsymbol{\delta}_z\frac{\partial\boldsymbol{\delta}_z}{\partial r}\tau_{zz} + \boldsymbol{\delta}_z\boldsymbol{\delta}_z\frac{\partial\tau_{zz}}{\partial r}\Bigg)\\
+ \frac{\boldsymbol{\delta}_\theta}{r}\cdot\Bigg(&\frac{\partial\boldsymbol{\delta}_r}{\partial\theta}\boldsymbol{\delta}_r\tau_{rr} + \boldsymbol{\delta}_r\frac{\partial\boldsymbol{\delta}_r}{\partial\theta}\tau_{rr} + \boldsymbol{\delta}_r\boldsymbol{\delta}_r\frac{\partial\tau_{rr}}{\partial\theta} + \frac{\partial\boldsymbol{\delta}_r}{\partial\theta}\boldsymbol{\delta}_\theta\tau_{r\theta} + \boldsymbol{\delta}_r\frac{\partial\boldsymbol{\delta}_\theta}{\partial\theta}\tau_{r\theta} + \boldsymbol{\delta}_r\boldsymbol{\delta}_\theta\frac{\partial\tau_{r\theta}}{\partial\theta}\\
&+ \frac{\partial\boldsymbol{\delta}_r}{\partial\theta}\boldsymbol{\delta}_z\tau_{rz} + \boldsymbol{\delta}_r\frac{\partial\boldsymbol{\delta}_z}{\partial\theta}\tau_{rz} + \boldsymbol{\delta}_r\boldsymbol{\delta}_z\frac{\partial\tau_{rz}}{\partial\theta} + \frac{\partial\boldsymbol{\delta}_\theta}{\partial\theta}\boldsymbol{\delta}_r\tau_{\theta r} + \boldsymbol{\delta}_\theta\frac{\partial\boldsymbol{\delta}_r}{\partial\theta}\tau_{\theta r} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_r\frac{\partial\tau_{\theta r}}{\partial\theta}\\
&+ \frac{\partial\boldsymbol{\delta}_\theta}{\partial\theta}\boldsymbol{\delta}_\theta\tau_{\theta\theta} + \boldsymbol{\delta}_\theta\frac{\partial\boldsymbol{\delta}_\theta}{\partial\theta}\tau_{\theta\theta} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta\frac{\partial\tau_{\theta\theta}}{\partial\theta} + \frac{\partial\boldsymbol{\delta}_\theta}{\partial\theta}\boldsymbol{\delta}_z\tau_{\theta z} + \boldsymbol{\delta}_\theta\frac{\partial\boldsymbol{\delta}_z}{\partial\theta}\tau_{\theta z} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_z\frac{\partial\tau_{\theta z}}{\partial\theta}\\
&+ \frac{\partial\boldsymbol{\delta}_z}{\partial\theta}\boldsymbol{\delta}_r\tau_{zr} + \boldsymbol{\delta}_z\frac{\partial\boldsymbol{\delta}_r}{\partial\theta}\tau_{zr} + \boldsymbol{\delta}_z\boldsymbol{\delta}_r\frac{\partial\tau_{zr}}{\partial\theta} + \frac{\partial\boldsymbol{\delta}_z}{\partial\theta}\boldsymbol{\delta}_\theta\tau_{z\theta} + \boldsymbol{\delta}_z\frac{\partial\boldsymbol{\delta}_\theta}{\partial\theta}\tau_{z\theta} + \boldsymbol{\delta}_z\boldsymbol{\delta}_\theta\frac{\partial\tau_{z\theta}}{\partial\theta}\\
&+ \frac{\partial\boldsymbol{\delta}_z}{\partial\theta}\boldsymbol{\delta}_z\tau_{zz} + \boldsymbol{\delta}_z\frac{\partial\boldsymbol{\delta}_z}{\partial\theta}\tau_{zz} + \boldsymbol{\delta}_z\boldsymbol{\delta}_z\frac{\partial\tau_{zz}}{\partial\theta}\Bigg)\\
+ \boldsymbol{\delta}_z\cdot\Bigg(&\frac{\partial\boldsymbol{\delta}_r}{\partial z}\boldsymbol{\delta}_r\tau_{rr} + \boldsymbol{\delta}_r\frac{\partial\boldsymbol{\delta}_r}{\partial z}\tau_{rr} + \boldsymbol{\delta}_r\boldsymbol{\delta}_r\frac{\partial\tau_{rr}}{\partial z} + \frac{\partial\boldsymbol{\delta}_r}{\partial z}\boldsymbol{\delta}_\theta\tau_{r\theta} + \boldsymbol{\delta}_r\frac{\partial\boldsymbol{\delta}_\theta}{\partial z}\tau_{r\theta} + \boldsymbol{\delta}_r\boldsymbol{\delta}_\theta\frac{\partial\tau_{r\theta}}{\partial z}\\
&+ \frac{\partial\boldsymbol{\delta}_z}{\partial z}\boldsymbol{\delta}_z\tau_{rz} + \boldsymbol{\delta}_r\frac{\partial\boldsymbol{\delta}_z}{\partial z}\tau_{rz} + \boldsymbol{\delta}_r\boldsymbol{\delta}_z\frac{\partial\tau_{rz}}{\partial z} + \frac{\partial\boldsymbol{\delta}_\theta}{\partial z}\boldsymbol{\delta}_r\tau_{\theta r} + \boldsymbol{\delta}_\theta\frac{\partial\boldsymbol{\delta}_r}{\partial z}\tau_{\theta r} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_r\frac{\partial\tau_{\theta r}}{\partial z}\\
&+ \frac{\partial\boldsymbol{\delta}_\theta}{\partial z}\boldsymbol{\delta}_\theta\tau_{\theta\theta} + \boldsymbol{\delta}_\theta\frac{\partial\boldsymbol{\delta}_\theta}{\partial z}\tau_{\theta\theta} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta\frac{\partial\tau_{\theta\theta}}{\partial z} + \frac{\partial\boldsymbol{\delta}_\theta}{\partial z}\boldsymbol{\delta}_z\tau_{\theta z} + \boldsymbol{\delta}_\theta\frac{\partial\boldsymbol{\delta}_z}{\partial z}\tau_{\theta z} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_z\frac{\partial\tau_{\theta z}}{\partial z}\\
&+ \frac{\partial\boldsymbol{\delta}_z}{\partial z}\boldsymbol{\delta}_r\tau_{zr} + \boldsymbol{\delta}_z\frac{\partial\boldsymbol{\delta}_r}{\partial z}\tau_{zr} + \boldsymbol{\delta}_z\boldsymbol{\delta}_r\frac{\partial\tau_{zr}}{\partial z} + \frac{\partial\boldsymbol{\delta}_z}{\partial z}\boldsymbol{\delta}_\theta\tau_{z\theta} + \boldsymbol{\delta}_z\frac{\partial\boldsymbol{\delta}_\theta}{\partial z}\tau_{z\theta} + \boldsymbol{\delta}_z\boldsymbol{\delta}_\theta\frac{\partial\tau_{z\theta}}{\partial z}\\
&+ \frac{\partial\boldsymbol{\delta}_z}{\partial z}\boldsymbol{\delta}_z\tau_{zz} + \boldsymbol{\delta}_z\frac{\partial\boldsymbol{\delta}_z}{\partial z}\tau_{zz} + \boldsymbol{\delta}_z\boldsymbol{\delta}_z\frac{\partial\tau_{zz}}{\partial z}\Bigg)\\
= \boldsymbol{\delta}_r\cdot\Bigg(&\boldsymbol{\delta}_r\boldsymbol{\delta}_r\frac{\partial\tau_{rr}}{\partial r} + \boldsymbol{\delta}_r\boldsymbol{\delta}_\theta\frac{\partial\tau_{r\theta}}{\partial r} + \boldsymbol{\delta}_r\boldsymbol{\delta}_z\frac{\partial\tau_{rz}}{\partial r} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_r\frac{\partial\tau_{\theta r}}{\partial r} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta\frac{\partial\tau_{\theta\theta}}{\partial r} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_z\frac{\partial\tau_{\theta z}}{\partial r} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_z\frac{\partial\tau_{\theta z}}{\partial r}\\
&+\boldsymbol{\delta}_z\boldsymbol{\delta}_r\frac{\partial\tau_{zr}}{\partial r} + \boldsymbol{\delta}_z\boldsymbol{\delta}_\theta\frac{\partial\tau_{z\theta}}{\partial r} + \boldsymbol{\delta}_z\boldsymbol{\delta}_z\frac{\partial\tau_{zz}}{\partial r}\Bigg)\\
+ \frac{\boldsymbol{\delta}_\theta}{r}\cdot\Bigg(&\boldsymbol{\delta}_\theta\boldsymbol{\delta}_r\tau_{rr} + \boldsymbol{\delta}_r\boldsymbol{\delta}_\theta\tau_{rr} + \boldsymbol{\delta}_r\boldsymbol{\delta}_r\frac{\partial\tau_{rr}}{\partial\theta} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta\tau_{r\theta} - \boldsymbol{\delta}_r\boldsymbol{\delta}_r\tau_{r\theta} + \boldsymbol{\delta}_r\boldsymbol{\delta}_\theta\frac{\partial\tau_{r\theta}}{\partial\theta} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_z\tau_{rz} + \boldsymbol{\delta}_r\boldsymbol{\delta}_z\frac{\partial\tau_{rz}}{\partial\theta}\\
&- \boldsymbol{\delta}_r\boldsymbol{\delta}_r\tau_{\theta r} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta\tau_{\theta r} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_r\frac{\partial\tau_{\theta r}}{\partial\theta} - \boldsymbol{\delta}_r\boldsymbol{\delta}_\theta\tau_{\theta\theta} - \boldsymbol{\delta}_\theta\boldsymbol{\delta}_r\tau_{\theta\theta} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta\frac{\partial\tau_{\theta\theta}}{\partial\theta} - \boldsymbol{\delta}_r\boldsymbol{\delta}_z\tau_{\theta z}\\
&+ \boldsymbol{\delta}_\theta\boldsymbol{\delta}_z\frac{\partial\tau_{\theta z}}{\partial\theta} + \boldsymbol{\delta}_z\boldsymbol{\delta}_\theta\tau_{zr} + \boldsymbol{\delta}_z\boldsymbol{\delta}_r\frac{\partial\tau_{zr}}{\partial\theta} - \boldsymbol{\delta}_z\boldsymbol{\delta}_r\tau_{z\theta} + \boldsymbol{\delta}_z\boldsymbol{\delta}_\theta\frac{\partial\tau_{z\theta}}{\partial\theta} + \boldsymbol{\delta}_z\boldsymbol{\delta}_z\frac{\partial\tau_{zz}}{\partial\theta}\Bigg)\\
+ \boldsymbol{\delta}_z\cdot\Bigg(&\boldsymbol{\delta}_r\boldsymbol{\delta}_r\frac{\partial\tau_{rr}}{\partial z} + \boldsymbol{\delta}_r\boldsymbol{\delta}_\theta\frac{\partial\tau_{r\theta}}{\partial z} + \boldsymbol{\delta}_r\boldsymbol{\delta}_z\frac{\partial\tau_{rz}}{\partial z} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_r\frac{\partial\tau_{\theta r}}{\partial z} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta\frac{\partial\tau_{\theta\theta}}{\partial z} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_z\frac{\partial\tau_{\theta z}}{\partial z} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_z\frac{\partial\tau_{\theta z}}{\partial z}\\
&+\boldsymbol{\delta}_z\boldsymbol{\delta}_r\frac{\partial\tau_{zr}}{\partial z} + \boldsymbol{\delta}_z\boldsymbol{\delta}_\theta\frac{\partial\tau_{z\theta}}{\partial z} + \boldsymbol{\delta}_z\boldsymbol{\delta}_z\frac{\partial\tau_{zz}}{\partial z}\Bigg),
\end{aligned}
$$

where we used equations A.7-1, A.7-2, and A.7-3 to simplify the derivatives of the unit vectors. Now proceed with evaluating the dot products. The unit vectors outside the parentheses dot the first unit vector in each dyad.

$$\begin{aligned}\nabla\cdot\boldsymbol{\tau} &= \boldsymbol{\delta}_r\frac{\partial\tau_{rr}}{\partial r}+\boldsymbol{\delta}_\theta\frac{\partial\tau_{r\theta}}{\partial r}+\boldsymbol{\delta}_z\frac{\partial\tau_{rz}}{\partial r}\\ &\quad+\frac{1}{r}\left(\boldsymbol{\delta}_r\tau_{rr}+\boldsymbol{\delta}_\theta\tau_{r\theta}+\boldsymbol{\delta}_z\tau_{rz}+\boldsymbol{\delta}_\theta\tau_{\theta r}+\boldsymbol{\delta}_r\frac{\partial\tau_{\theta r}}{\partial\theta}-\boldsymbol{\delta}_r\tau_{\theta\theta}+\boldsymbol{\delta}_\theta\frac{\partial\tau_{\theta\theta}}{\partial\theta}+\boldsymbol{\delta}_z\frac{\partial\tau_{\theta z}}{\partial\theta}\right)\\ &\quad+\boldsymbol{\delta}_r\frac{\partial\tau_{zr}}{\partial z}+\boldsymbol{\delta}_\theta\frac{\partial\tau_{z\theta}}{\partial z}+\boldsymbol{\delta}_z\frac{\partial\tau_{zz}}{\partial z}\end{aligned}$$

Factor each of the unit vectors.

$$\begin{aligned}&=\boldsymbol{\delta}_r\left(\frac{\partial\tau_{rr}}{\partial r}+\frac{\tau_{rr}}{r}+\frac{1}{r}\frac{\partial\tau_{\theta r}}{\partial\theta}-\frac{\tau_{\theta\theta}}{r}+\frac{\partial\tau_{zr}}{\partial z}\right)\\ &\quad+\boldsymbol{\delta}_\theta\left(\frac{\partial\tau_{r\theta}}{\partial r}+\frac{\tau_{r\theta}}{r}+\frac{\tau_{\theta r}}{r}+\frac{1}{r}\frac{\partial\tau_{\theta\theta}}{\partial\theta}+\frac{\partial\tau_{z\theta}}{\partial z}\right)\\ &\quad+\boldsymbol{\delta}_z\left(\frac{\partial\tau_{rz}}{\partial r}+\frac{\tau_{rz}}{r}+\frac{1}{r}\frac{\partial\tau_{\theta z}}{\partial\theta}+\frac{\partial\tau_{zz}}{\partial z}\right)\\ &=\boldsymbol{\delta}_r\left[\frac{1}{r}\left(r\frac{\partial\tau_{rr}}{\partial r}+\tau_{rr}\right)+\frac{1}{r}\left(\frac{\partial\tau_{\theta r}}{\partial\theta}-\tau_{\theta\theta}\right)+\frac{\partial\tau_{zr}}{\partial z}\right]\\ &\quad+\boldsymbol{\delta}_\theta\left(\frac{\partial\tau_{r\theta}}{\partial r}+2\frac{\tau_{r\theta}}{r}-\frac{\tau_{r\theta}}{r}+\frac{\tau_{\theta r}}{r}+\frac{1}{r}\frac{\partial\tau_{\theta\theta}}{\partial\theta}+\frac{\partial\tau_{z\theta}}{\partial z}\right)\\ &\quad+\boldsymbol{\delta}_z\left[\frac{1}{r}\left(r\frac{\partial\tau_{rz}}{\partial r}+\tau_{rz}\right)+\frac{1}{r}\frac{\partial\tau_{\theta z}}{\partial\theta}+\frac{\partial\tau_{zz}}{\partial z}\right]\end{aligned}$$

Use the product rule to write the formula compactly.

$$\begin{aligned}&=\boldsymbol{\delta}_r\left[\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr})+\frac{1}{r}\left(\frac{\partial\tau_{\theta r}}{\partial\theta}-\tau_{\theta\theta}\right)+\frac{\partial\tau_{zr}}{\partial z}\right]\\ &\quad+\boldsymbol{\delta}_\theta\left[\frac{1}{r^2}\left(r^2\frac{\partial\tau_{r\theta}}{\partial r}+2r\tau_{r\theta}\right)+\frac{\tau_{\theta r}-\tau_{r\theta}}{r}+\frac{1}{r}\frac{\partial\tau_{\theta\theta}}{\partial\theta}+\frac{\partial\tau_{z\theta}}{\partial z}\right]\\ &\quad+\boldsymbol{\delta}_z\left[\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz})+\frac{1}{r}\frac{\partial\tau_{\theta z}}{\partial\theta}+\frac{\partial\tau_{zz}}{\partial z}\right]\end{aligned}$$

Therefore, the divergence of $\boldsymbol{\tau}$ in cylindrical coordinates is

$$\begin{aligned}\nabla\cdot\boldsymbol{\tau}&=\boldsymbol{\delta}_r\left[\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr})+\frac{1}{r}\left(\frac{\partial\tau_{\theta r}}{\partial\theta}-\tau_{\theta\theta}\right)+\frac{\partial\tau_{zr}}{\partial z}\right]\\ &\quad+\boldsymbol{\delta}_\theta\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta})+\frac{\tau_{\theta r}-\tau_{r\theta}}{r}+\frac{1}{r}\frac{\partial\tau_{\theta\theta}}{\partial\theta}+\frac{\partial\tau_{z\theta}}{\partial z}\right]\\ &\quad+\boldsymbol{\delta}_z\left[\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz})+\frac{1}{r}\frac{\partial\tau_{\theta z}}{\partial\theta}+\frac{\partial\tau_{zz}}{\partial z}\right].\end{aligned}$$