Exercise 5

A constant force $\mathbf{F}$ acts on a body moving with a velocity $\mathbf{v}$, which is not necessarily collinear with $\mathbf{F}$. Show that the rate at which $\mathbf{F}$ does work on the body is $W = (\mathbf{F} \cdot \mathbf{v})$.

Solution

Work is defined as

$$W = \mathbf{F} \cdot \mathbf{x},$$

where $\mathbf{x}$ is the position vector of the body. To obtain the rate that work is done with respect to time, take the derivative of both sides with respect to t.

$$\frac{dW}{dt} = \frac{d}{dt}(\mathbf{F} \cdot \mathbf{x})$$

Expand the right-hand side.

$$\frac{dW}{dt} = \frac{d\mathbf{F}}{dt} \cdot \mathbf{x} + \mathbf{F} \cdot \frac{d\mathbf{x}}{dt}$$

Since $\mathbf{F}$ is constant, $d\mathbf{F}/dt = 0$, and we have

$$\frac{dW}{dt} = \mathbf{F} \cdot \frac{d\mathbf{x}}{dt}.$$

The rate of change of the position vector with respect to time is the velocity $\mathbf{v}$. Therefore,

$$\frac{dW}{dt} = \mathbf{F} \cdot \mathbf{v}.$$