Exercise 6

Explain carefully the statement after Eq. A.2-21 that the ith component of $[\mathbf{v} \times \mathbf{w}]$ is $\sum_j \sum_k \varepsilon_{ijk} v_j w_k$.

Solution

If we have two vectors, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$, then the cross product is

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \boldsymbol{\delta}_1 & \boldsymbol{\delta}_2 & \boldsymbol{\delta}_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

The ith component is

$$\begin{aligned} &v_2 w_3 - v_3 w_2 \quad \text{when } i = 1 \\ &v_3 w_1 - v_1 w_3 \quad \text{when } i = 2 \\ &v_1 w_2 - v_2 w_1 \quad \text{when } i = 3. \end{aligned}$$

Since the permutation symbol ε_{ijk} is defined as

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } ijk = 123, 231, \text{ or } 312 \\ -1 & \text{if } ijk = 321, 132, \text{ or } 213 \\ 0 & \text{if any indices are the same} \end{cases},$$

the ith component can be written as

$$\sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} v_j w_k.$$

We will prove this now. Expanding the double sum, we have

$$\begin{aligned} \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} v_j w_k = \sum_{j=1}^{3} (\varepsilon_{ij1} v_j w_1 + \varepsilon_{ij2} v_j w_2 + \varepsilon_{ij3} v_j w_3) &= \sum_{j=1}^{3} \varepsilon_{ij1} v_j w_1 + \sum_{j=1}^{3} \varepsilon_{ij2} v_j w_2 + \sum_{j=1}^{3} \varepsilon_{ij3} v_j w_3 \\ &= \underbrace{\varepsilon_{i11} v_1 w_1}_{=0} + \varepsilon_{i21} v_2 w_1 + \varepsilon_{i31} v_3 w_1 \\ &\quad + \varepsilon_{i12} v_1 w_2 + \underbrace{\varepsilon_{i22} v_2 w_2}_{=0} + \varepsilon_{i32} v_3 w_2 \\ &\quad + \varepsilon_{i13} v_1 w_3 + \varepsilon_{i23} v_2 w_3 + \underbrace{\varepsilon_{i33} v_3 w_3}_{=0}. \end{aligned}$$

When $i = 1$, we have

$$\begin{aligned} \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{1jk} v_j w_k &= \underbrace{\varepsilon_{121} v_2 w_1}_{=0} + \underbrace{\varepsilon_{131} v_3 w_1}_{=0} + \underbrace{\varepsilon_{112} v_1 w_2}_{=0} + \varepsilon_{132} v_3 w_2 + \underbrace{\varepsilon_{113} v_1 w_3}_{=0} + \varepsilon_{123} v_2 w_3 \\ &= v_2 w_3 - v_3 w_2. \end{aligned}$$

When $i = 2$, we have

$$\begin{aligned}\sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon_{2jk}v_j w_k &= \underbrace{\varepsilon_{221}v_2 w_1}_{=0} + \varepsilon_{231}v_3 w_1 + \underbrace{\varepsilon_{212}v_1 w_2}_{=0} + \underbrace{\varepsilon_{232}v_3 w_2}_{=0} + \varepsilon_{213}v_1 w_3 + \underbrace{\varepsilon_{223}v_2 w_3}_{=0} \\ &= v_3 w_1 - v_1 w_3.\end{aligned}$$

When $i = 3$, we have

$$\begin{aligned}\sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon_{3jk}v_j w_k &= \varepsilon_{321}v_2 w_1 + \underbrace{\varepsilon_{331}v_3 w_1}_{=0} + \varepsilon_{312}v_1 w_2 + \underbrace{\varepsilon_{332}v_3 w_2}_{=0} + \underbrace{\varepsilon_{313}v_1 w_3}_{=0} + \underbrace{\varepsilon_{323}v_2 w_3}_{=0} \\ &= v_1 w_2 - v_2 w_1.\end{aligned}$$

Therefore, the ith component of $\mathbf{v} \times \mathbf{w}$ can be written as

$$\sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon_{ijk}v_j w_k.$$