Exercise 5

Consider a rigid structure composed of point particles joined by massless rods. The particles are numbered $1, 2, 3, \ldots, N$, and the particle masses are m_v ($v = 1, 2, \ldots, N$). The locations of the particles with respect to the center of mass are $\mathbf{R}_v$. The entire structure rotates on an axis passing through the center of mass with an angular velocity $\mathbf{W}$. Show that the angular momentum with respect to the center of mass is

$$\mathbf{L} = \sum_v m_v[\mathbf{R}_v \times [\mathbf{W} \times \mathbf{R}_v]]$$

Then show that the latter expression may be rewritten as

$$\mathbf{L} = [\boldsymbol{\Phi} \cdot \mathbf{W}]$$

where

$$\boldsymbol{\Phi} = \sum_v m_v \left\{(\mathbf{R}_v \cdot \mathbf{R}_v)\boldsymbol{\delta} - \mathbf{R}_v\mathbf{R}_v\right\}$$

is the *moment-of-inertia tensor*.

Solution

The angular momentum of mass v is the cross product of its position vector with its linear momentum vector.

$$\mathbf{L}_v = \mathbf{R}_v \times \mathbf{p}_v$$

The linear momentum is mass times linear velocity.

$$\mathbf{L}_v = \mathbf{R}_v \times [m_v\mathbf{v}_v]$$

m_v is a constant, so it can be brought in front.

$$\mathbf{L}_v = m_v[\mathbf{R}_v \times \mathbf{v}_v]$$

The linear velocity is the cross product of angular velocity and the position vector.

$$\mathbf{L}_v = m_v[\mathbf{R}_v \times [\mathbf{W} \times \mathbf{R}_v]]$$

To obtain the total angular momentum, we have to sum the angular momentum of each mass in the rigid structure.

$$\mathbf{L} = \sum_{v=1}^{N} \mathbf{L}_v$$

Therefore,

$$\mathbf{L} = \sum_{v=1}^{N} m_v[\mathbf{R}_v \times [\mathbf{W} \times \mathbf{R}_v]].$$

In order to write this in the desired form, make use of the so-called BAC-CAB vector identity.

$$\mathbf{A} \times [\mathbf{B} \times \mathbf{C}] = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

Doing so gives us

$$\mathbf{L} = \sum_{v=1}^{N} m_v[\mathbf{W}(\mathbf{R}_v \cdot \mathbf{R}_v) - \mathbf{R}_v(\mathbf{R}_v \cdot \mathbf{W})].$$

Move $\mathbf{W}$ to the right side of the term.

$$\mathbf{L} = \sum_{v=1}^{N} m_v[(\mathbf{R}_v \cdot \mathbf{R}_v)\mathbf{W} - \mathbf{R}_v(\mathbf{R}_v \cdot \mathbf{W})]$$

We can write $\mathbf{W}$ in terms of the unit tensor $\boldsymbol{\delta}$ as $\boldsymbol{\delta} \cdot \mathbf{W}$. This will be shown now.

$$\begin{aligned}\boldsymbol{\delta} \cdot \mathbf{W} = \left(\sum_{i=1}^{3}\sum_{j=1}^{3} \boldsymbol{\delta}_i\boldsymbol{\delta}_j\delta_{ij}\right) \cdot \left(\sum_{k=1}^{3} \boldsymbol{\delta}_k W_k\right) = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_i(\boldsymbol{\delta}_j \cdot \boldsymbol{\delta}_k)\delta_{ij}W_k &= \sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_j(\boldsymbol{\delta}_j \cdot \boldsymbol{\delta}_k)W_k \\ &= \sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_j\delta_{jk}W_k \\ &= \sum_{k=1}^{3} \boldsymbol{\delta}_k W_k \\ &= \mathbf{W}\end{aligned}$$

Hence,

$$\mathbf{L} = \sum_{v=1}^{N} m_v[(\mathbf{R}_v \cdot \mathbf{R}_v)[\boldsymbol{\delta} \cdot \mathbf{W}] - \mathbf{R}_v(\mathbf{R}_v \cdot \mathbf{W})].$$

Factor out $\mathbf{W}$.

$$\mathbf{L} = \sum_{v=1}^{N} m_v\{(\mathbf{R}_v \cdot \mathbf{R}_v)\boldsymbol{\delta} - \mathbf{R}_v\mathbf{R}_v\} \cdot \mathbf{W}$$

Therefore,

$$\mathbf{L} = \boldsymbol{\Phi} \cdot \mathbf{W},$$

where

$$\boldsymbol{\Phi} = \sum_{v=1}^{N} m_v \left\{(\mathbf{R}_v \cdot \mathbf{R}_v)\boldsymbol{\delta} - \mathbf{R}_v\mathbf{R}_v\right\}$$

is the moment-of-inertia tensor.