Exercise 1

Perform all the operations in Eq. A.4-6 by writing out all the summations instead of using the $\sum$ notation.

Solution

Eq. A.4-6 gives the formula for the divergence of a vector field $\mathbf{v}$. Without sums, it becomes

$$\begin{aligned}
\nabla \cdot \mathbf{v} &= \left(\boldsymbol{\delta}_1 \frac{\partial}{\partial x_1} + \boldsymbol{\delta}_2 \frac{\partial}{\partial x_2} + \boldsymbol{\delta}_3 \frac{\partial}{\partial x_3}\right) \cdot (\boldsymbol{\delta}_1 v_1 + \boldsymbol{\delta}_2 v_2 + \boldsymbol{\delta}_3 v_3) \\
&= \boldsymbol{\delta}_1 \frac{\partial}{\partial x_1} \cdot (\boldsymbol{\delta}_1 v_1 + \boldsymbol{\delta}_2 v_2 + \boldsymbol{\delta}_3 v_3) + \boldsymbol{\delta}_2 \frac{\partial}{\partial x_2} \cdot (\boldsymbol{\delta}_1 v_1 + \boldsymbol{\delta}_2 v_2 + \boldsymbol{\delta}_3 v_3) + \boldsymbol{\delta}_3 \frac{\partial}{\partial x_3} \cdot (\boldsymbol{\delta}_1 v_1 + \boldsymbol{\delta}_2 v_2 + \boldsymbol{\delta}_3 v_3) \\
&= (\boldsymbol{\delta}_1 \cdot \boldsymbol{\delta}_1) \frac{\partial v_1}{\partial x_1} + (\boldsymbol{\delta}_1 \cdot \boldsymbol{\delta}_2) \frac{\partial v_2}{\partial x_1} + (\boldsymbol{\delta}_1 \cdot \boldsymbol{\delta}_3) \frac{\partial v_3}{\partial x_1} + (\boldsymbol{\delta}_2 \cdot \boldsymbol{\delta}_1) \frac{\partial v_1}{\partial x_2} + (\boldsymbol{\delta}_2 \cdot \boldsymbol{\delta}_2) \frac{\partial v_2}{\partial x_2} + (\boldsymbol{\delta}_2 \cdot \boldsymbol{\delta}_3) \frac{\partial v_3}{\partial x_2} \\
&\qquad + (\boldsymbol{\delta}_3 \cdot \boldsymbol{\delta}_1) \frac{\partial v_1}{\partial x_3} + (\boldsymbol{\delta}_3 \cdot \boldsymbol{\delta}_2) \frac{\partial v_2}{\partial x_3} + (\boldsymbol{\delta}_3 \cdot \boldsymbol{\delta}_3) \frac{\partial v_3}{\partial x_3} \\
&= \delta_{11} \frac{\partial v_1}{\partial x_1} + \delta_{12} \frac{\partial v_2}{\partial x_1} + \delta_{13} \frac{\partial v_3}{\partial x_1} + \delta_{21} \frac{\partial v_1}{\partial x_2} + \delta_{22} \frac{\partial v_2}{\partial x_2} + \delta_{23} \frac{\partial v_3}{\partial x_2} + \delta_{31} \frac{\partial v_1}{\partial x_3} + \delta_{32} \frac{\partial v_2}{\partial x_3} + \delta_{33} \frac{\partial v_3}{\partial x_3} \\
&= \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}
\end{aligned}$$