Exercise 6

Verify that $(\nabla \cdot [\nabla \times \mathbf{v}]) = 0$ and $[\nabla \times \nabla s] = 0$.

[TYPO: 0 should be in bold, as the curl operator yields a vector, not a scalar.]

Solution

The First Vector Identity

$$\begin{aligned}
\nabla \cdot [\nabla \times \mathbf{v}] &= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left[\left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial}{\partial x_j}\right) \times \left(\sum_{k=1}^{3} \boldsymbol{\delta}_k v_k\right)\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left[\sum_{j=1}^{3}\sum_{k=1}^{3} (\boldsymbol{\delta}_j \times \boldsymbol{\delta}_k) \frac{\partial v_k}{\partial x_j}\right] = \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left(\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_l \varepsilon_{jkl} \frac{\partial v_k}{\partial x_j}\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_l) \varepsilon_{jkl} \frac{\partial}{\partial x_i}\left(\frac{\partial v_k}{\partial x_j}\right) = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \delta_{il} \varepsilon_{jkl} \frac{\partial}{\partial x_i}\left(\frac{\partial v_k}{\partial x_j}\right) \\
&= \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \varepsilon_{jkl} \frac{\partial}{\partial x_l}\left(\frac{\partial v_k}{\partial x_j}\right)
\end{aligned}$$

Rewrite the triple sum with j for l and l for j. This can be done because they are dummy indices.

$$= \sum_{l=1}^{3}\sum_{k=1}^{3}\sum_{j=1}^{3} \varepsilon_{lkj} \frac{\partial}{\partial x_j}\left(\frac{\partial v_k}{\partial x_l}\right)$$

Arrange the sums so that they're the same as the triple sum in blue. This can be done because the limits on each sum are constants.

$$= \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \varepsilon_{lkj} \frac{\partial}{\partial x_j}\left(\frac{\partial v_k}{\partial x_l}\right)$$

Use Clairaut's theorem to interchange the order of the derivatives. This is possible if the second derivatives are continuous.

$$= \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \varepsilon_{lkj} \frac{\partial}{\partial x_l}\left(\frac{\partial v_k}{\partial x_j}\right)$$

Move the j-index from the end to the beginning in the permutation symbol. Doing so does not change the sign.

$$= \sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \varepsilon_{jlk} \frac{\partial}{\partial x_l}\left(\frac{\partial v_k}{\partial x_j}\right)$$

Switch the k and l indices in the permutation symbol. Doing so changes the sign.

$$= -\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \varepsilon_{jkl} \frac{\partial}{\partial x_l}\left(\frac{\partial v_k}{\partial x_j}\right)$$

The only number equal to its negative is zero. Therefore,

$$\nabla \cdot [\nabla \times \mathbf{v}] = 0.$$

The Second Vector Identity

$$\nabla \times \nabla s = \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial s}{\partial x_j}\right) = \sum_{i=1}^{3} \sum_{j=1}^{3} (\boldsymbol{\delta}_i \times \boldsymbol{\delta}_j) \frac{\partial}{\partial x_i} \left(\frac{\partial s}{\partial x_j}\right)$$

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{ijk} \frac{\partial}{\partial x_i} \left(\frac{\partial s}{\partial x_j}\right)$$

Rewrite the triple sum with j for i and i for j. This can be done because they are dummy indices.

$$= \sum_{j=1}^{3} \sum_{i=1}^{3} \sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{jik} \frac{\partial}{\partial x_j} \left(\frac{\partial s}{\partial x_i}\right)$$

Arrange the sums so that they're the same as the triple sum in blue. This can be done because the limits on each sum are constants.

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{jik} \frac{\partial}{\partial x_j} \left(\frac{\partial s}{\partial x_i}\right)$$

Use Clairaut's theorem to interchange the order of the derivatives. This is possible if the second derivatives are continuous.

$$= \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{jik} \frac{\partial}{\partial x_i} \left(\frac{\partial s}{\partial x_j}\right)$$

Switch the j and i indices in the permutation symbol. Doing so changes the sign.

$$= -\sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{ijk} \frac{\partial}{\partial x_i} \left(\frac{\partial s}{\partial x_j}\right)$$

The only vector equal to its negative is the zero vector. Therefore,

$$\nabla \times \nabla s = \mathbf{0}.$$