Problem 1D.1

Uniform rotation of a fluid.

(a) Verify that the velocity distribution in a fluid in a state of pure rotation (i.e., rotating as a rigid body) is $\mathbf{v} = [\mathbf{w} \times \mathbf{r}]$, where $\mathbf{w}$ is the angular velocity (a constant) and $\mathbf{r}$ is the position vector, with components x, y, z.

(b) What are $\nabla\mathbf{v} + (\nabla\mathbf{v})^\dagger$ and $(\nabla \cdot \mathbf{v})$ for the flow field in (a)?

(c) Interpret Eq. 1.2-7 in terms of the results in (b).

Solution

Part (a)

Choose the xyz-coordinate system such that the positive z-axis points in the same direction as the angular velocity vector $\mathbf{w}$, $\mathbf{w} = w\boldsymbol{\delta}_z$. In Cartesian coordinates the position vector is $\mathbf{r} = x\boldsymbol{\delta}_x + y\boldsymbol{\delta}_y + z\boldsymbol{\delta}_z$. Evaluate the cross product now.

$$\mathbf{v} = \mathbf{w} \times \mathbf{r} = \begin{vmatrix} \boldsymbol{\delta}_x & \boldsymbol{\delta}_y & \boldsymbol{\delta}_z \\ 0 & 0 & w \\ x & y & z \end{vmatrix} = -wy\boldsymbol{\delta}_x + wx\boldsymbol{\delta}_y$$

If the fluid is in a state of pure rotation, then in polar coordinates we expect the velocity vector to only have a θ-component, $\mathbf{v} = v_\theta\boldsymbol{\delta}_\theta$. Use the variable transformations,

$$x = r\cos\theta$$
$$y = r\sin\theta,$$

and use the unit vector transformations (from Appendix A.6 on page 827),

$$\boldsymbol{\delta}_x = \cos\theta\boldsymbol{\delta}_r - \sin\theta\boldsymbol{\delta}_\theta + 0\boldsymbol{\delta}_z$$
$$\boldsymbol{\delta}_y = \sin\theta\boldsymbol{\delta}_r + \cos\theta\boldsymbol{\delta}_\theta + 0\boldsymbol{\delta}_z$$
$$\boldsymbol{\delta}_z = 0\boldsymbol{\delta}_r + 0\boldsymbol{\delta}_\theta + 1\boldsymbol{\delta}_z.$$

Making these substitutions, the velocity becomes

$$\mathbf{v} = -wr\sin\theta(\cos\theta\boldsymbol{\delta}_r - \sin\theta\boldsymbol{\delta}_\theta) + wr\cos\theta(\sin\theta\boldsymbol{\delta}_r + \cos\theta\boldsymbol{\delta}_\theta).$$

Expand the result and factor the polar unit vectors.

$$\mathbf{v} = (\cancel{-wr\sin\theta\cos\theta} + \cancel{wr\sin\theta\cos\theta})\boldsymbol{\delta}_r + (wr\sin^2\theta + wr\cos^2\theta)\boldsymbol{\delta}_\theta$$

Apply $\sin^2\theta + \cos^2\theta = 1$.

$$\mathbf{v} = wr\boldsymbol{\delta}_\theta$$

Since the velocity only has a θ-component, the fluid is in pure rotation as expected.

Part (b)

We will calculate $\nabla\mathbf{v}$, then $\nabla\mathbf{v}^\dagger$, and then take the sum to find $\nabla\mathbf{v} + \nabla\mathbf{v}^\dagger$. Use $\mathbf{v} = -wy\boldsymbol{\delta}_x + wx\boldsymbol{\delta}_y$ from part (a).

$$\begin{aligned}
\nabla\mathbf{v} &= \left(\boldsymbol{\delta}_x\frac{\partial}{\partial x} + \boldsymbol{\delta}_y\frac{\partial}{\partial y} + \boldsymbol{\delta}_z\frac{\partial}{\partial z}\right)(-wy\boldsymbol{\delta}_x + wx\boldsymbol{\delta}_y) \\
&= \boldsymbol{\delta}_x\frac{\partial}{\partial x}(-wy\boldsymbol{\delta}_x + wx\boldsymbol{\delta}_y) \\
&\quad + \boldsymbol{\delta}_y\frac{\partial}{\partial y}(-wy\boldsymbol{\delta}_x + wx\boldsymbol{\delta}_y) \\
&\quad + \boldsymbol{\delta}_z\frac{\partial}{\partial z}(-wy\boldsymbol{\delta}_x + wx\boldsymbol{\delta}_y) \\
&= \boldsymbol{\delta}_x\boldsymbol{\delta}_x\frac{\partial}{\partial x}(-wy) + \boldsymbol{\delta}_x\boldsymbol{\delta}_y\frac{\partial}{\partial x}(wx) \\
&\quad + \boldsymbol{\delta}_y\boldsymbol{\delta}_x\frac{\partial}{\partial y}(-wy) + \boldsymbol{\delta}_y\boldsymbol{\delta}_y\frac{\partial}{\partial y}(wx) \\
&\quad + \boldsymbol{\delta}_z\boldsymbol{\delta}_x\frac{\partial}{\partial z}(-wy) + \boldsymbol{\delta}_z\boldsymbol{\delta}_y\frac{\partial}{\partial z}(wx) \\
&= w\boldsymbol{\delta}_x\boldsymbol{\delta}_y - w\boldsymbol{\delta}_y\boldsymbol{\delta}_x
\end{aligned}$$

From this we can write down the transpose. The component of $\boldsymbol{\delta}_y\boldsymbol{\delta}_x$ switches with that of $\boldsymbol{\delta}_x\boldsymbol{\delta}_y$.

$$\nabla\mathbf{v}^\dagger = -w\boldsymbol{\delta}_x\boldsymbol{\delta}_y + w\boldsymbol{\delta}_y\boldsymbol{\delta}_x$$

Adding $\nabla\mathbf{v}$ and $\nabla\mathbf{v}^\dagger$, we get

$$\nabla\mathbf{v} + \nabla\mathbf{v}^\dagger = (w - w)\boldsymbol{\delta}_x\boldsymbol{\delta}_y + (-w + w)\boldsymbol{\delta}_y\boldsymbol{\delta}_x.$$

Therefore,

$$\nabla\mathbf{v} + \nabla\mathbf{v}^\dagger = \mathbf{0}.$$

The divergence of $\mathbf{v}$ is

$$\nabla\cdot\mathbf{v} = \frac{\partial}{\partial x}(-wy) + \frac{\partial}{\partial y}(wx) + \frac{\partial}{\partial z}(0).$$

Therefore,

$$\nabla\cdot\mathbf{v} = 0.$$

Part (c)

Eq. 1.2-7 in the text gives the formula for the viscous stress tensor $\boldsymbol{\tau}$.

$$\boldsymbol{\tau} = -\mu(\nabla\mathbf{v} + (\nabla\mathbf{v})^\dagger) + \left(\frac{2}{3}\mu - \kappa\right)(\nabla\cdot\mathbf{v})\boldsymbol{\delta} \tag{1.2-7}$$

From the results in part (b), we obtain

$$\boldsymbol{\tau} = \mathbf{0}$$

for a fluid in pure rotation. This implies that there are no viscous forces in the fluid.