Problem 1D.2

Force on a surface of arbitrary orientation.[5] (Fig. 1D.2) Consider the material within an element of volume $OABC$ that is in a state of equilibrium, so that the sum of the forces acting on the triangular faces $\triangle OBC$, $\triangle OCA$, $\triangle OAB$, and $\triangle ABC$ must be zero. Let the area of $\triangle ABC$ be dS, and the force per unit area acting from the minus to the plus side of dS be the vector $\boldsymbol{\pi}_n$. Show that $\boldsymbol{\pi}_n = [\mathbf{n} \cdot \boldsymbol{\pi}]$.

(a) Show that the area of $\triangle OBC$ is the same as the area of the projection $\triangle ABC$ on the yz-plane; this is $(\mathbf{n} \cdot \boldsymbol{\delta}_x)dS$. Write similar expressions for the areas of $\triangle OCA$ and $\triangle OAB$.

(b) Show that according to Table 1.2-1 the force per unit area on $\triangle OBC$ is $\boldsymbol{\delta}_x \pi_{xx} + \boldsymbol{\delta}_y \pi_{xy} + \boldsymbol{\delta}_z \pi_{xz}$. Write similar force expressions for $\triangle OCA$ and $\triangle OAB$.

(c) Show that the force balance for the volume element $OABC$ gives

$$\boldsymbol{\pi}_n = \sum_i \sum_j (\mathbf{n} \cdot \boldsymbol{\delta}_i)(\boldsymbol{\delta}_j \pi_{ij}) = [\mathbf{n} \cdot \sum_i \sum_j \boldsymbol{\delta}_i \boldsymbol{\delta}_j \pi_{ij}] \tag{1D.2-1}$$

in which the indices i, j take on the values x, y, z. The double sum in the last expression is the stress tensor $\boldsymbol{\pi}$ written as a sum of products of unit dyads and components.

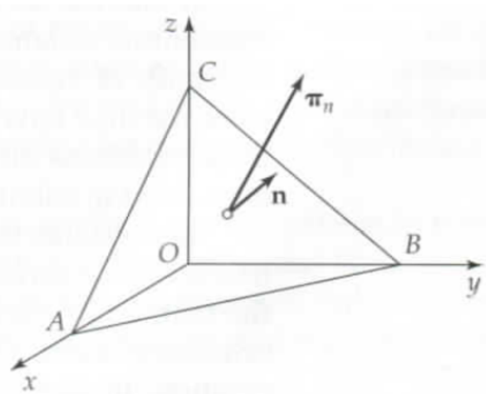

Fig. 1D.2 Element of volume $OABC$ over which a force balance is made. The vector $\boldsymbol{\pi}_n = [\mathbf{n} \cdot \boldsymbol{\pi}]$ is the force per unit area exerted by the minus material (material inside $OABC$) on the plus material (material outside $OABC$). The vector $\mathbf{n}$ is the outwardly directed unit normal vector on face ABC.

Solution

[5] M. Abraham and R. Becker, *The Classical Theory of Electricity and Magnetism*, Blackie and Sons, London (1952), pp. 44–45.

Part (a)

The trick here is to treat area as a vector—let the magnitude be the area and let the direction be the unit vector perpendicular to the area. Taking the dot product of the area vector of $\triangle ABC$ with a unit vector in some direction will give us the projection of the area of $\triangle ABC$ onto the plane that that unit vector is perpendicular to.

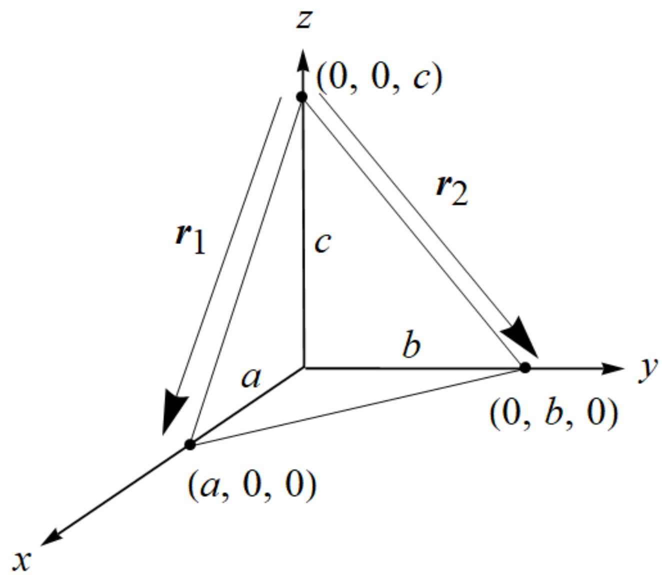

Figure 1: This figure shows the dimensions of volume $OABC$ and the position vectors, $\mathbf{r}_1$ and $\mathbf{r}_2$.

Face ABC is a triangle with sides $\sqrt{a^2+b^2}$, $\sqrt{b^2+c^2}$, and $\sqrt{a^2+c^2}$, so the area of it is given by Hero's formula,

$$A_{ABC} = \sqrt{s(s-\sqrt{a^2+b^2})(s-\sqrt{b^2+c^2})(s-\sqrt{a^2+c^2})},$$

where s is the sum of the sides divided by 2. Simplifying the result, we obtain

$$A_{ABC} = \frac{1}{2}\sqrt{b^2c^2+a^2c^2+a^2b^2}.$$

To obtain a unit vector normal to $\triangle ABC$, write expressions for the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ in Figure 1 by considering the position vectors of the points A, B, and C.

$$\mathbf{r}_1 = \langle a,0,0\rangle - \langle 0,0,c\rangle = \langle a,0,-c\rangle$$
$$\mathbf{r}_2 = \langle 0,b,0\rangle - \langle 0,0,c\rangle = \langle 0,b,-c\rangle$$

Take the cross product of these vectors to find a vector perpendicular to ABC.

$$\mathbf{r}_1 \times \mathbf{r}_2 = \begin{vmatrix} \boldsymbol{\delta}_x & \boldsymbol{\delta}_y & \boldsymbol{\delta}_z \\ a & 0 & -c \\ 0 & b & -c \end{vmatrix} = bc\boldsymbol{\delta}_x + ac\boldsymbol{\delta}_y + ab\boldsymbol{\delta}_z$$

To make this a unit vector $\mathbf{n}$, divide the vector by its magnitude.

$$\mathbf{n} = \frac{bc\boldsymbol{\delta}_x + ac\boldsymbol{\delta}_y + ab\boldsymbol{\delta}_z}{\sqrt{b^2c^2+a^2c^2+a^2b^2}}$$

Thus, the vector representing the area of $\triangle ABC$ is

$$\mathbf{A}_{ABC} = \frac{1}{2}\sqrt{b^2c^2 + a^2c^2 + a^2b^2}\,\frac{bc\boldsymbol{\delta}_x + ac\boldsymbol{\delta}_y + ab\boldsymbol{\delta}_z}{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}} = \frac{1}{2}\langle bc, ac, ab\rangle.$$

The unit vector perpendicular to $\triangle OBC$ is $\boldsymbol{\delta}_x$, so the area of $\triangle OBC$ is given by

$$A_{OBC} = \mathbf{A}_{ABC} \cdot \boldsymbol{\delta}_x = \frac{1}{2}bc.$$

The unit vector perpendicular to $\triangle OAC$ is $\boldsymbol{\delta}_y$, so the area of $\triangle OAC$ is given by

$$A_{OAC} = \mathbf{A}_{ABC} \cdot \boldsymbol{\delta}_y = \frac{1}{2}ac.$$

The unit vector perpendicular to $\triangle OAB$ is $\boldsymbol{\delta}_z$, so the area of $\triangle OAB$ is given by

$$A_{OAB} = \mathbf{A}_{ABC} \cdot \boldsymbol{\delta}_z = \frac{1}{2}ab.$$

As faces OBC, OAC, and OAB are triangles, these results are indeed correct.

Part (b)

The molecular momentum stress tensor $\boldsymbol{\pi}$ gives us the force per unit area.

$$\begin{aligned}\boldsymbol{\pi} = {} & \boldsymbol{\delta}_x\boldsymbol{\delta}_x\pi_{xx} + \boldsymbol{\delta}_x\boldsymbol{\delta}_y\pi_{xy} + \boldsymbol{\delta}_x\boldsymbol{\delta}_z\pi_{xz} \\ & + \boldsymbol{\delta}_y\boldsymbol{\delta}_x\pi_{yx} + \boldsymbol{\delta}_y\boldsymbol{\delta}_y\pi_{yy} + \boldsymbol{\delta}_y\boldsymbol{\delta}_z\pi_{yz} \\ & + \boldsymbol{\delta}_z\boldsymbol{\delta}_x\pi_{zx} + \boldsymbol{\delta}_z\boldsymbol{\delta}_y\pi_{zy} + \boldsymbol{\delta}_z\boldsymbol{\delta}_z\pi_{zz}\end{aligned}$$

The unit vector perpendicular to $\triangle OBC$ is $\boldsymbol{\delta}_x$, so the force per unit area on $\triangle OBC$ is $\boldsymbol{\pi}_x = \boldsymbol{\delta}_x \cdot \boldsymbol{\pi}$.

$$\begin{aligned}\boldsymbol{\delta}_x \cdot \boldsymbol{\pi} &= \boldsymbol{\delta}_x \cdot (\boldsymbol{\delta}_x\boldsymbol{\delta}_x\pi_{xx} + \boldsymbol{\delta}_x\boldsymbol{\delta}_y\pi_{xy} + \boldsymbol{\delta}_x\boldsymbol{\delta}_z\pi_{xz} \\ &\qquad + \boldsymbol{\delta}_y\boldsymbol{\delta}_x\pi_{yx} + \boldsymbol{\delta}_y\boldsymbol{\delta}_y\pi_{yy} + \boldsymbol{\delta}_y\boldsymbol{\delta}_z\pi_{yz} \\ &\qquad + \boldsymbol{\delta}_z\boldsymbol{\delta}_x\pi_{zx} + \boldsymbol{\delta}_z\boldsymbol{\delta}_y\pi_{zy} + \boldsymbol{\delta}_z\boldsymbol{\delta}_z\pi_{zz}) \\ &= (\boldsymbol{\delta}_x \cdot \boldsymbol{\delta}_x)\boldsymbol{\delta}_x\pi_{xx} + (\boldsymbol{\delta}_x \cdot \boldsymbol{\delta}_x)\boldsymbol{\delta}_y\pi_{xy} + (\boldsymbol{\delta}_x \cdot \boldsymbol{\delta}_x)\boldsymbol{\delta}_z\pi_{xz} \\ &\qquad + (\boldsymbol{\delta}_x \cdot \boldsymbol{\delta}_y)\boldsymbol{\delta}_x\pi_{yx} + (\boldsymbol{\delta}_x \cdot \boldsymbol{\delta}_y)\boldsymbol{\delta}_y\pi_{yy} + (\boldsymbol{\delta}_x \cdot \boldsymbol{\delta}_y)\boldsymbol{\delta}_z\pi_{yz} \\ &\qquad + (\boldsymbol{\delta}_x \cdot \boldsymbol{\delta}_z)\boldsymbol{\delta}_x\pi_{zx} + (\boldsymbol{\delta}_x \cdot \boldsymbol{\delta}_z)\boldsymbol{\delta}_y\pi_{zy} + (\boldsymbol{\delta}_x \cdot \boldsymbol{\delta}_z)\boldsymbol{\delta}_z\pi_{zz} \\ &= \boldsymbol{\delta}_x\pi_{xx} + \boldsymbol{\delta}_y\pi_{xy} + \boldsymbol{\delta}_z\pi_{xz}\end{aligned}$$

The unit vector perpendicular to $\triangle OAC$ is $\boldsymbol{\delta}_y$, so the force per unit area on $\triangle OAC$ is $\boldsymbol{\pi}_y = \boldsymbol{\delta}_y \cdot \boldsymbol{\pi}$.

$$\begin{aligned}\boldsymbol{\delta}_y \cdot \boldsymbol{\pi} &= \boldsymbol{\delta}_y \cdot (\boldsymbol{\delta}_x\boldsymbol{\delta}_x\pi_{xx} + \boldsymbol{\delta}_x\boldsymbol{\delta}_y\pi_{xy} + \boldsymbol{\delta}_x\boldsymbol{\delta}_z\pi_{xz} \\ &\qquad + \boldsymbol{\delta}_y\boldsymbol{\delta}_x\pi_{yx} + \boldsymbol{\delta}_y\boldsymbol{\delta}_y\pi_{yy} + \boldsymbol{\delta}_y\boldsymbol{\delta}_z\pi_{yz} \\ &\qquad + \boldsymbol{\delta}_z\boldsymbol{\delta}_x\pi_{zx} + \boldsymbol{\delta}_z\boldsymbol{\delta}_y\pi_{zy} + \boldsymbol{\delta}_z\boldsymbol{\delta}_z\pi_{zz}) \\ &= (\boldsymbol{\delta}_y \cdot \boldsymbol{\delta}_x)\boldsymbol{\delta}_x\pi_{xx} + (\boldsymbol{\delta}_y \cdot \boldsymbol{\delta}_x)\boldsymbol{\delta}_y\pi_{xy} + (\boldsymbol{\delta}_y \cdot \boldsymbol{\delta}_x)\boldsymbol{\delta}_z\pi_{xz} \\ &\qquad + (\boldsymbol{\delta}_y \cdot \boldsymbol{\delta}_y)\boldsymbol{\delta}_x\pi_{yx} + (\boldsymbol{\delta}_y \cdot \boldsymbol{\delta}_y)\boldsymbol{\delta}_y\pi_{yy} + (\boldsymbol{\delta}_y \cdot \boldsymbol{\delta}_y)\boldsymbol{\delta}_z\pi_{yz} \\ &\qquad + (\boldsymbol{\delta}_y \cdot \boldsymbol{\delta}_z)\boldsymbol{\delta}_x\pi_{zx} + (\boldsymbol{\delta}_y \cdot \boldsymbol{\delta}_z)\boldsymbol{\delta}_y\pi_{zy} + (\boldsymbol{\delta}_y \cdot \boldsymbol{\delta}_z)\boldsymbol{\delta}_z\pi_{zz} \\ &= \boldsymbol{\delta}_x\pi_{yx} + \boldsymbol{\delta}_y\pi_{yy} + \boldsymbol{\delta}_z\pi_{yz}\end{aligned}$$

The unit vector perpendicular to $\triangle OAB$ is $\boldsymbol{\delta}_z$, so the force per unit area on $\triangle OAB$ is $\boldsymbol{\pi}_z = \boldsymbol{\delta}_z \cdot \boldsymbol{\pi}$.

$$\begin{aligned}
\boldsymbol{\delta}_z \cdot \boldsymbol{\pi} &= \boldsymbol{\delta}_z \cdot (\boldsymbol{\delta}_x\boldsymbol{\delta}_x\pi_{xx} + \boldsymbol{\delta}_x\boldsymbol{\delta}_y\pi_{xy} + \boldsymbol{\delta}_x\boldsymbol{\delta}_z\pi_{xz} \\
&\qquad + \boldsymbol{\delta}_y\boldsymbol{\delta}_x\pi_{yx} + \boldsymbol{\delta}_y\boldsymbol{\delta}_y\pi_{yy} + \boldsymbol{\delta}_y\boldsymbol{\delta}_z\pi_{yz} \\
&\qquad + \boldsymbol{\delta}_z\boldsymbol{\delta}_x\pi_{zx} + \boldsymbol{\delta}_z\boldsymbol{\delta}_y\pi_{zy} + \boldsymbol{\delta}_z\boldsymbol{\delta}_z\pi_{zz}) \\
&= (\boldsymbol{\delta}_z \cdot \boldsymbol{\delta}_x)\boldsymbol{\delta}_x\pi_{xx} + (\boldsymbol{\delta}_z \cdot \boldsymbol{\delta}_x)\boldsymbol{\delta}_y\pi_{xy} + (\boldsymbol{\delta}_z \cdot \boldsymbol{\delta}_x)\boldsymbol{\delta}_z\pi_{xz} \\
&\qquad + (\boldsymbol{\delta}_z \cdot \boldsymbol{\delta}_y)\boldsymbol{\delta}_x\pi_{yx} + (\boldsymbol{\delta}_z \cdot \boldsymbol{\delta}_y)\boldsymbol{\delta}_y\pi_{yy} + (\boldsymbol{\delta}_z \cdot \boldsymbol{\delta}_y)\boldsymbol{\delta}_z\pi_{yz} \\
&\qquad + (\boldsymbol{\delta}_z \cdot \boldsymbol{\delta}_z)\boldsymbol{\delta}_x\pi_{zx} + (\boldsymbol{\delta}_z \cdot \boldsymbol{\delta}_z)\boldsymbol{\delta}_y\pi_{zy} + (\boldsymbol{\delta}_z \cdot \boldsymbol{\delta}_z)\boldsymbol{\delta}_z\pi_{zz} \\
&= \boldsymbol{\delta}_x\pi_{zx} + \boldsymbol{\delta}_y\pi_{zy} + \boldsymbol{\delta}_z\pi_{zz}
\end{aligned}$$

Do note that the order we take the dot product is important. The reason we choose to do $\boldsymbol{\delta}_x \cdot \boldsymbol{\pi}$ rather than $\boldsymbol{\pi} \cdot \boldsymbol{\delta}_x$ is so that the first index of π becomes x rather than the second index. This is because the first index indicates what direction is perpendicular to the area. The second one indicates the direction the force is acting. In part (a) it didn't matter whether we did $\mathbf{A}_{ABC} \cdot \boldsymbol{\delta}_x$ or $\boldsymbol{\delta}_x \cdot \mathbf{A}_{ABC}$ since $\boldsymbol{\delta}_x$ and $\mathbf{A}_{ABC}$ were both vectors. Here, though, $\boldsymbol{\pi}$ is a second-order tensor, so the dot product is not commutative.

Part (c)

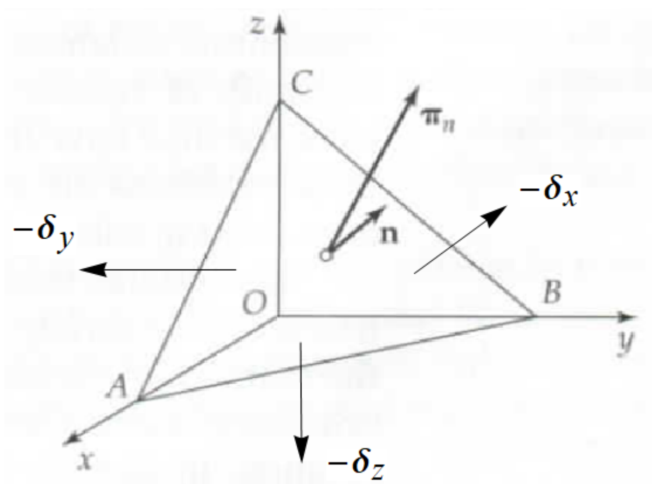

Figure 2: Fig. 1D.2 with added outward normal unit vectors for faces OCB, OAB, and OAC.

According to Newton's second law, if the volume element $OABC$ is in equilibrium, then the sum of the forces acting on its faces must be equal to the zero vector.

$$\sum_i \mathbf{F}_i = \mathbf{0}$$

There are four different forces to consider: those acting on $\triangle OBC$ in the x-direction, those acting on $\triangle OAC$ in the y-direction, those acting on $\triangle OAB$ in the z-direction, and those acting on $\triangle ABC$ in the n-direction. Since the outward normal vectors of faces OBC, OAC, and OAB point in the negative directions, there are minus signs in front of these forces.

$$-\mathbf{F}_x - \mathbf{F}_y - \mathbf{F}_z + \mathbf{F}_n = \mathbf{0}$$

Since the molecular momentum stress tensor $\boldsymbol{\pi}$ represents force per unit area, we can get the force by multiplying it by the area.

$$-\boldsymbol{\pi}_x A_{OBC} - \boldsymbol{\pi}_y A_{OAC} - \boldsymbol{\pi}_z A_{OAB} + \boldsymbol{\pi}_n \cdot A_{ABC} = \mathbf{0}$$

Use the formulas for the areas of the triangles from part (a).

$$-\boldsymbol{\pi}_x \left(\frac{1}{2}bc\right) - \boldsymbol{\pi}_y \left(\frac{1}{2}ac\right) - \boldsymbol{\pi}_z \left(\frac{1}{2}ab\right) + \boldsymbol{\pi}_n \left(\frac{1}{2}\sqrt{b^2c^2 + a^2c^2 + a^2b^2}\right) = \mathbf{0}$$

Use the expressions in part (b) for $\boldsymbol{\pi}_x$, $\boldsymbol{\pi}_y$, and $\boldsymbol{\pi}_z$.

$$-\boldsymbol{\delta}_x \cdot \boldsymbol{\pi} \left(\frac{1}{2}bc\right) - \boldsymbol{\delta}_y \cdot \boldsymbol{\pi} \left(\frac{1}{2}ac\right) - \boldsymbol{\delta}_z \cdot \boldsymbol{\pi} \left(\frac{1}{2}ab\right) + \boldsymbol{\pi}_n \left(\frac{1}{2}\sqrt{b^2c^2 + a^2c^2 + a^2b^2}\right) = \mathbf{0}$$

Use the expressions in part (b) for $\boldsymbol{\delta}_x \cdot \boldsymbol{\pi}$, $\boldsymbol{\delta}_y \cdot \boldsymbol{\pi}$, and $\boldsymbol{\delta}_z \cdot \boldsymbol{\pi}$.

$$\begin{aligned} &-(\boldsymbol{\delta}_x \pi_{xx} + \boldsymbol{\delta}_y \pi_{xy} + \boldsymbol{\delta}_z \pi_{xz})\left(\frac{1}{2}bc\right) - (\boldsymbol{\delta}_x \pi_{yx} + \boldsymbol{\delta}_y \pi_{yy} + \boldsymbol{\delta}_z \pi_{yz})\left(\frac{1}{2}ac\right) \\ &\qquad - (\boldsymbol{\delta}_x \pi_{zx} + \boldsymbol{\delta}_y \pi_{zy} + \boldsymbol{\delta}_z \pi_{zz})\left(\frac{1}{2}ab\right) + \boldsymbol{\pi}_n \left(\frac{1}{2}\sqrt{b^2c^2 + a^2c^2 + a^2b^2}\right) = \mathbf{0} \end{aligned}$$

Multiply both sides by 2 and factor the unit vectors.

$$\begin{aligned} &-\boldsymbol{\delta}_x(bc\pi_{xx} + ac\pi_{yx} + ab\pi_{zx}) - \boldsymbol{\delta}_y(bc\pi_{xy} + ac\pi_{yy} + ab\pi_{zy}) \\ &\qquad - \boldsymbol{\delta}_z(bc\pi_{xz} + ac\pi_{yz} + ab\pi_{zz}) + \boldsymbol{\pi}_n\sqrt{b^2c^2 + a^2c^2 + a^2b^2} = \mathbf{0} \end{aligned}$$

Solve this equation for $\boldsymbol{\pi}_n$.

$$\boldsymbol{\pi}_n = \frac{\boldsymbol{\delta}_x(bc\pi_{xx} + ac\pi_{yx} + ab\pi_{zx}) + \boldsymbol{\delta}_y(bc\pi_{xy} + ac\pi_{yy} + ab\pi_{zy}) + \boldsymbol{\delta}_z(bc\pi_{xz} + ac\pi_{yz} + ab\pi_{zz})}{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}}$$

Distribute the square root term.

$$\begin{aligned} \boldsymbol{\pi}_n = \boldsymbol{\delta}_x &\left(\frac{bc}{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}}\pi_{xx} + \frac{ac}{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}}\pi_{yx} + \frac{ab}{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}}\pi_{zx}\right) \\ + \boldsymbol{\delta}_y &\left(\frac{bc}{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}}\pi_{xy} + \frac{ac}{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}}\pi_{yy} + \frac{ab}{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}}\pi_{zy}\right) \\ + \boldsymbol{\delta}_z &\left(\frac{bc}{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}}\pi_{xz} + \frac{ac}{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}}\pi_{yz} + \frac{ab}{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}}\pi_{zz}\right) \end{aligned}$$

From part (a) we have

$$\mathbf{n} = \frac{bc\boldsymbol{\delta}_x + ac\boldsymbol{\delta}_y + ab\boldsymbol{\delta}_z}{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}},$$

so

$$\boldsymbol{\pi}_n = \boldsymbol{\delta}_x(n_x\pi_{xx} + n_y\pi_{yx} + n_z\pi_{zx}) + \boldsymbol{\delta}_y(n_x\pi_{xy} + n_y\pi_{yy} + n_z\pi_{zy}) + \boldsymbol{\delta}_z(n_x\pi_{xz} + n_y\pi_{yz} + n_z\pi_{zz}).$$

This can be written compactly using sums.

$$\boldsymbol{\pi}_n = \boldsymbol{\delta}_x\left(\sum_{i=1}^{3} n_i\pi_{ix}\right) + \boldsymbol{\delta}_y\left(\sum_{i=1}^{3} n_i\pi_{iy}\right) + \boldsymbol{\delta}_z\left(\sum_{i=1}^{3} n_i\pi_{iz}\right)$$

Factor out n_i and the sum.

$$\boldsymbol{\pi}_n = \sum_{i=1}^{3} n_i(\boldsymbol{\delta}_x \pi_{ix} + \boldsymbol{\delta}_y \pi_{iy} + \boldsymbol{\delta}_z \pi_{iz})$$

This can be written even more compactly by introducing another sum.

$$\begin{aligned}
\boldsymbol{\pi}_n &= \sum_{i=1}^{3} n_i \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \pi_{ij} \right) \\
&= \sum_{i=1}^{3} \sum_{j=1}^{3} n_i \boldsymbol{\delta}_j \pi_{ij} \\
&= \sum_{i=1}^{3} \sum_{j=1}^{3} (\mathbf{n} \cdot \boldsymbol{\delta}_i) \boldsymbol{\delta}_j \pi_{ij} \\
&= \mathbf{n} \cdot \sum_{i=1}^{3} \sum_{j=1}^{3} \boldsymbol{\delta}_i \boldsymbol{\delta}_j \pi_{ij}
\end{aligned}$$

Therefore,

$$\boldsymbol{\pi}_n = \mathbf{n} \cdot \boldsymbol{\pi}.$$