Problem 2B.11

The cone-and-plate viscometer (see Fig. 2B.11). A cone-and-plate viscometer consists of a flat plate and an inverted cone, whose apex just contacts the plate. The liquid whose viscosity is to be measured is placed in the gap between the cone and plate. The cone is rotated at a known angular velocity Ω, and the torque T_z required to turn the cone is measured. Find an expression for the viscosity of the fluid in terms of Ω, T_z, and the angle ψ_0 between the cone and the plate. For commercial instruments ψ_0 is about 1 degree.

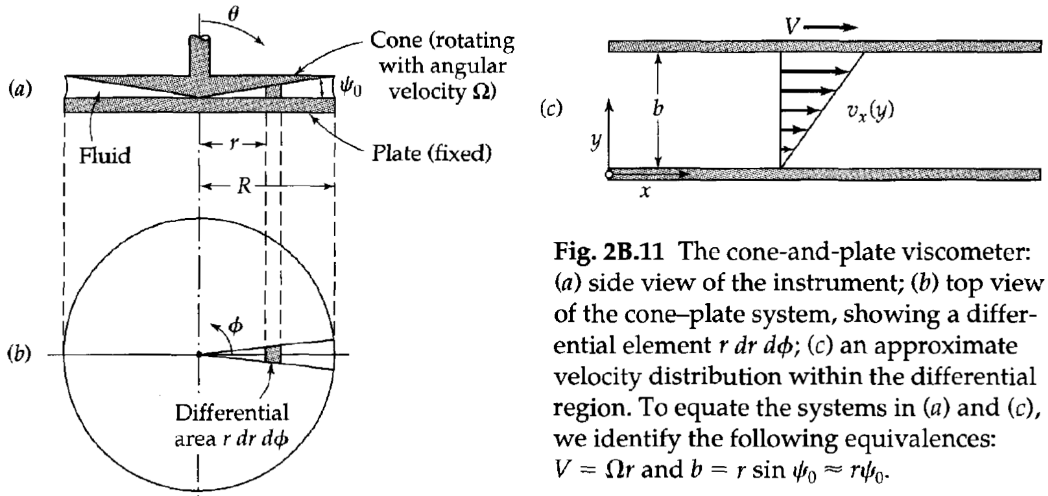


Fig. 2B.11 The cone-and-plate viscometer: (*a*) side view of the instrument; (*b*) top view of the cone–plate system, showing a differential element $r\,dr\,d\phi$; (*c*) an approximate velocity distribution within the differential region. To equate the systems in (*a*) and (*c*), we identify the following equivalences: $V = \Omega r$ and $b = r \sin\psi_0 \approx r\psi_0$.

(a) Assume that locally the velocity distribution in the gap can be very closely approximated by that for flow between parallel plates, the upper one moving with a constant speed. Verify that this leads to the *approximate* velocity distribution (in spherical coordinates)

$$\frac{v_\phi}{r} = \Omega\left(\frac{(\pi/2)-\theta}{\psi_0}\right) \tag{2B.11-1}$$

This approximation should be rather good, because ψ_0 is so small.

(b) From the velocity distribution in Eq. 2B.11-1 and Appendix B.1, show that a reasonable expression for the shear stress is

$$\tau_{\theta\phi} = \mu(\Omega/\psi_0) \tag{2B.11-2}$$

This result shows that the shear stress is uniform throughout the gap. It is this fact that makes the cone-and-plate viscometer quite attractive. The instrument is widely used, particularly in the polymer industry.

(c) Show that the torque required to turn the cone is given by

$$T_z = \frac{2}{3}\pi\mu\Omega R^3/\psi_0 \tag{2B.11-3}$$

This is the standard formula for calculating the viscosity from measurements of the torque and angular velocity for a cone-plate assembly with known R and ψ_0.

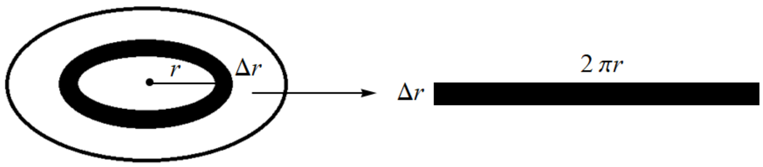
(d) For a cone-and-plate instrument with radius 10 cm and angle ψ_0 equal to 0.5 degree, what torque (in $\text{dyn}\cdot\text{cm}$) is required to turn the cone at an angular velocity of 10 radians per minute if the fluid viscosity is 100 cp?

Answer: **(d)** $40,000$ dyn $\cdot$ cm

Solution

Part (a)

Because the linear velocity of the cone's top varies with r ($V = \Omega r$), we can treat each slice of the circle with circumference $2\pi r$ and thickness Δr as a linear element with length $L = 2\pi r$ and thickness Δr moving at speed V. The problem essentially boils down to Couette flow, where a



fluid flows in a slit and one wall moves while the other remains stationary. This is illustrated in part (c) of Fig. 2B.11. Thus, a rectangular shell will be considered to determine the velocity distribution. We assume that the fluid flows in the x-direction and that its velocity varies in the y-direction.

$$v_x = v_x(y)$$

Consequently, only ϕ_{yx} (the x-momentum in the positive y-direction) and ϕ_{xx} (the x-momentum in the positive x-direction) contribute to the momentum balance. We also assume that the flow is only due to the spinning top and is not a result of gravity or a pressure difference.

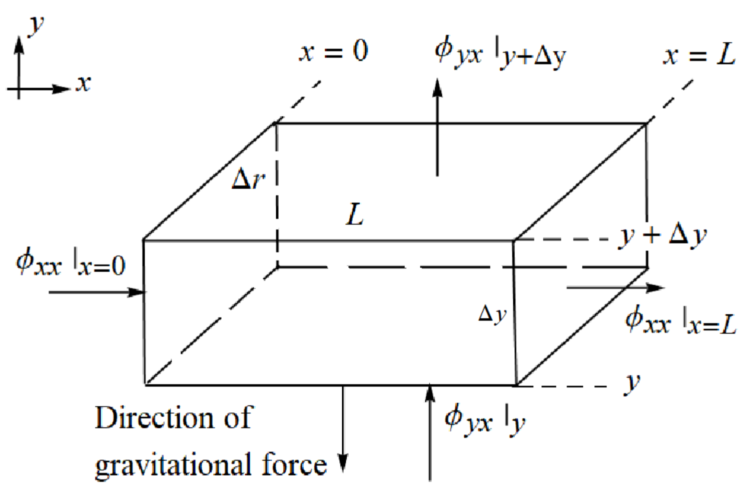


Figure 1: This is the shell over which the momentum balance is made for the flow in a slit.

Rate of x-momentum into the shell at $x = 0$:	$(\Delta r \Delta y)\phi_{xx}\|_{x=0}$
Rate of x-momentum into the shell at $x = L$:	$(\Delta r \Delta y)\phi_{xx}\|_{x=L}$
Rate of x-momentum into the shell at y:	$(L\Delta r)\phi_{yx}\|_{y}$
Rate of x-momentum into the shell at $y + \Delta y$:	$(L\Delta r)\phi_{yx}\|_{y+\Delta y}$
Component of gravitational force on the shell in x-direction:	0

If we assume steady flow, then the momentum balance is

$$\text{Rate of momentum in} - \text{Rate of momentum out} + \text{Force of gravity} = \mathbf{0}.$$

Considering only the x-component, we have

$$(\Delta r \Delta y)\phi_{xx}|_{x=0} - (\Delta r \Delta y)\phi_{xx}|_{x=L} + (L\Delta r)\phi_{yx}|_{y} - (L\Delta r)\phi_{yx}|_{y+\Delta y} = 0.$$

Factor the left side.

$$-\Delta r \Delta y(\phi_{xx}|_{x=L} - \phi_{xx}|_{x=0}) - L\Delta r(\phi_{yx}|_{y+\Delta y} - \phi_{yx}|_{y}) = 0$$

Divide both sides by $L\Delta r \Delta y$.

$$-\frac{\phi_{xx}|_{x=L} - \phi_{xx}|_{x=0}}{L} - \frac{\phi_{yx}|_{y+\Delta y} - \phi_{yx}|_{y}}{\Delta y} = 0$$

Take the limit as $\Delta y \to 0$.

$$-\frac{\phi_{xx}|_{x=L} - \phi_{xx}|_{x=0}}{L} - \lim_{\Delta y \to 0} \frac{\phi_{yx}|_{y+\Delta y} - \phi_{yx}|_{y}}{\Delta y} = 0$$

The second term is the first derivative of ϕ_{yx} with respect to y.

$$-\frac{\phi_{xx}|_{x=L} - \phi_{xx}|_{x=0}}{L} - \frac{d\phi_{yx}}{dy} = 0$$

Now substitute the expressions for ϕ_{yx} and ϕ_{xx}.

$$\phi_{yx} = \tau_{yx} + \cancel{\rho v_y v_x} = \tau_{yx}$$
$$\phi_{xx} = \cancel{p\delta_{xx}} + \cancel{\tau_{xx}} + \rho v_x v_x = \rho v_x^2$$

Since v_x does not depend on x, the ρv_x^2 terms cancel and we get

$$-\frac{\cancel{\phi_{xx}|_{x=L}} - \cancel{\phi_{xx}|_{x=0}}}{L} - \frac{d\tau_{yx}}{dy} = 0$$

or

$$\frac{d\tau_{yx}}{dy} = 0.$$

From Newton's law of viscosity, $\tau_{yx} = -\mu(dv_x/dy)$, so we have

$$\frac{d}{dy}\left(-\mu \frac{dv_x}{dy}\right) = 0.$$

Viscosity is assumed to be constant, so we can pull it and the minus sign in front of the derivative. Divide both sides by $-\mu$.

$$\frac{d^2 v_x}{dy^2} = 0$$

Using the coordinate system in part (c) of Fig. 2B.11, the boundary conditions are as follows.

$$\text{B.C. 1:} \quad v_x = 0 \quad \text{when} \quad y = 0$$
$$\text{B.C. 2:} \quad v_x = V \quad \text{when} \quad y = b$$

Integrate the differential equation with respect to y.

$$\frac{dv_x}{dy} = C_1$$

Integrate the differential equation with respect to y once more.

$$v_x(y) = C_1 y + C_2$$

Apply the two boundary conditions now.

$$v_x(0) = C_1(0) + C_2 = 0$$
$$v_x(b) = C_1(b) + C_2 = V$$

Solving this system of equations, we get $C_1 = V/b$ and $C_2 = 0$. Hence, we have for the velocity distribution

$$v_x(y) = \frac{V}{b}y.$$

Now we can change back to rotational flow in spherical coordinates; that is, the fluid actually flows in the ϕ direction and its velocity varies with r [$v_\phi = v_\phi(r)$]. We have $V = \Omega r$, $b = r \sin \psi_0 \approx r\psi_0$, and $y \approx r(\pi/2 - \theta)$.

$$v_\phi = \frac{\Omega \cancel{r}}{\cancel{r}\psi_0} r \left(\frac{\pi}{2} - \theta\right)$$

Therefore,

$$\frac{v_\phi}{r} = \Omega \left(\frac{(\pi/2) - \theta}{\psi_0}\right).$$

Part (b)

The shear stress on a surface element of the cone by the fluid is given by $\tau_{\theta\phi}$ (the θ-direction is perpendicular to the surface element, and the shear is in the ϕ-direction). According to Appendix B.1 on page 844,

$$\tau_{\theta\phi} = -\mu \left[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}\right].$$

Since the velocity is in the ϕ-direction only, the second term in the square brackets is zero. Also, θ is a constant on the cone's surface ($\theta = \theta_0$), so the formula for $\tau_{\theta\phi}$ simplifies considerably.

$$\begin{aligned}
\tau_{\theta\phi} &= -\mu \frac{\sin\theta_0}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\phi}{\sin\theta_0}\right) \\
&= -\mu \frac{1}{r}\frac{\partial v_\phi}{\partial\theta} \\
&= -\frac{\mu}{r}\frac{\partial}{\partial\theta}\left[\Omega r \left(\frac{\frac{\pi}{2} - \theta}{\psi_0}\right)\right] \\
&= -\frac{\mu}{r}\left[\Omega r \left(-\frac{1}{\psi_0}\right)\right]
\end{aligned}$$

Therefore,

$$\tau_{\theta\phi} = \mu \frac{\Omega}{\psi_0}.$$

Part (c)

Since the shearing force is in the ϕ-direction, it is perpendicular to the radial direction. The torque then is just the product of this force with the moment arm.

$$T_z = rF$$

To get the shearing force, we multiply (integrate) the shear stress by the area of the cone that the fluid is in contact with.

$$T_z = \int r \cdot \underbrace{\tau_{\theta\phi}|_{\theta=\frac{\pi}{2}-\psi_0}\, dA}_{\text{force}}$$

Since the fluid is at a lower value of θ than the cone's surface at $\theta = \frac{\pi}{2} - \psi_0$, no minus sign is needed in front of $\tau_{\theta\phi}$. Looking at part (b) in Fig. 2B.11, we see that the area differential is $dA = r\, dr\, d\phi$.

$$\begin{aligned} T_z &= \int_0^{2\pi}\int_0^R r \cdot \tau_{\theta\phi}|_{\theta=\frac{\pi}{2}-\psi_0}\, r\, dr\, d\phi \\ &= \int_0^{2\pi}\int_0^R r^2 \left(\mu\frac{\Omega}{\psi_0}\right) dr\, d\phi \\ &= \mu\frac{\Omega}{\psi_0}\left(\int_0^{2\pi} d\phi\right)\left(\int_0^R r^2\, dr\right) \\ &= \mu\frac{\Omega}{\psi_0}(2\pi)\left(\frac{R^3}{3}\right) \end{aligned}$$

Therefore,

$$T_z = \frac{2}{3}\pi\mu\Omega\frac{R^3}{\psi_0}.$$

Part (d)

We have the following values for the variables. The conversion factor for Pa to dyn/cm^2 is found in Table F.3-2 on page 869.

$$\begin{aligned} R &= 10 \text{ cm} \\ \psi_0 &= 0.5^\circ \times \frac{\pi \text{ rad}}{180^\circ} = \frac{\pi}{360} \text{ rad} \\ \Omega &= 10\, \frac{\text{rad}}{\cancel{\text{min}}} \times \frac{1\, \cancel{\text{min}}}{60 \text{ s}} = \frac{1}{6}\, \frac{\text{rad}}{\text{s}} \\ \mu &= 100\, \cancel{\text{cp}} \times \frac{10^{-3}\, \cancel{\text{Pa}} \cdot \text{s}}{1\, \cancel{\text{cp}}} \times \frac{10\, \frac{\text{dyn}}{\text{cm}^2}}{1\, \cancel{\text{Pa}}} = 1\, \frac{\text{dyn}}{\text{cm}^2} \cdot \text{s} \end{aligned}$$

Plugging the numbers into the formula for torque T_z, we get

$$\begin{aligned} T_z &= \frac{2}{3}\pi\mu\Omega\frac{R^3}{\psi_0} \\ &= \frac{2}{3}\pi\left(1\, \frac{\text{dyn}}{\text{cm}^2} \cdot \cancel{\text{s}}\right)\left(\frac{1}{6}\, \frac{\cancel{\text{rad}}}{\cancel{\text{s}}}\right)\frac{(10 \text{ cm})^3}{\frac{\pi}{360}\, \cancel{\text{rad}}} \\ &= 40,000\, \frac{\text{dyn}}{\text{cm}^2} \cdot \text{cm}^3 = 40,000 \text{ dyn} \cdot \text{cm}. \end{aligned}$$