Problem 2B.7

Annular flow with inner cylinder moving axially (see Fig. 2B.7). A cylindrical rod of radius κR moves axially with velocity $v_z = v_0$ along the axis of a cylindrical cavity of radius R as seen in the figure. The pressure at both ends of the cavity is the same, so that the fluid moves through the annular region solely because of the rod motion.

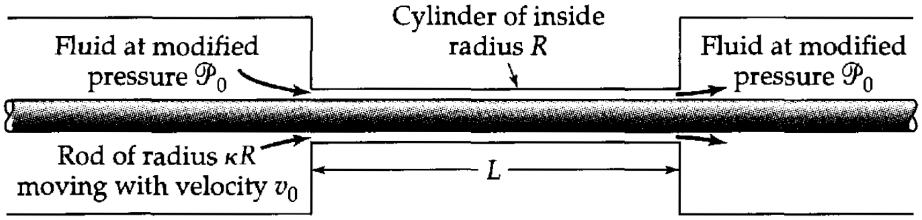

Fig. 2B.7 Annular flow with the inner cylinder moving axially.

(a) Find the velocity distribution in the narrow annular region.

(b) Find the mass rate of flow through the annular region.

(c) Obtain the viscous force acting on the rod over the length L.

(d) Show that the result in (c) can be written as a "plane slit" formula multiplied by a "curvature correction." Problems of this kind arise in studying the performance of wire-coating dies.[1]

Answers: **(a)** $\dfrac{v_z}{v_0} = \dfrac{\ln(r/R)}{\ln \kappa}$

(b) $w = \dfrac{\pi R^2 v_0 \rho}{2}\left[\dfrac{(1-\kappa^2)}{\ln(1/\kappa)} - 2\kappa^2\right]$

(c) $F_z = -2\pi L \mu v_0 / \ln(1/\kappa)$

(d) $F_z = \dfrac{-2\pi L \mu v_0}{\varepsilon}\left(1 - \dfrac{1}{2}\varepsilon - \dfrac{1}{12}\varepsilon^2 + \cdots\right)$ where $\varepsilon = 1 - \kappa$ (see Problem 2B.5)

Solution

Unlike the previous problem, the fluid here is flowing horizontally, so gravity will not influence its velocity. If we assume no-slip boundary conditions, then the fluid velocity at the cylindrical rod $r = \kappa R$ is v_0 and the fluid velocity at the outer wall $r = R$ is 0.

Part (a)

Choose a cylindrical coordinate system with the positive z-axis pointing to the right, the direction the inner cylinder is moving in. Then the fluid flows in the z-direction and varies as a function of radius from the cylinder's axis.

$$v_z = v_z(r)$$

[1] J. B. Paton, P. H. Squires, W. H. Darnell, F. M. Cash, and J. F. Carley, *Processing of Thermoplastic Materials*, E. C. Bernhardt (ed.), Reinhold, New York (1959), Chapter 4.

As a result, the boundary conditions are written as follows.

$$\text{B.C. 1:} \quad v_z = v_0 \quad \text{when} \quad r = \kappa R$$
$$\text{B.C. 2:} \quad v_z = 0 \quad \text{when} \quad r = R$$

Gravity is pointing down, so we can say that the pressure does not depend on z.

$$p \neq p(z)$$

Because $v_z = v_z(r)$, only ϕ_{rz} (the z-momentum in the positive r-direction) and ϕ_{zz} (the z-momentum in the positive z-direction) contribute to the momentum balance.

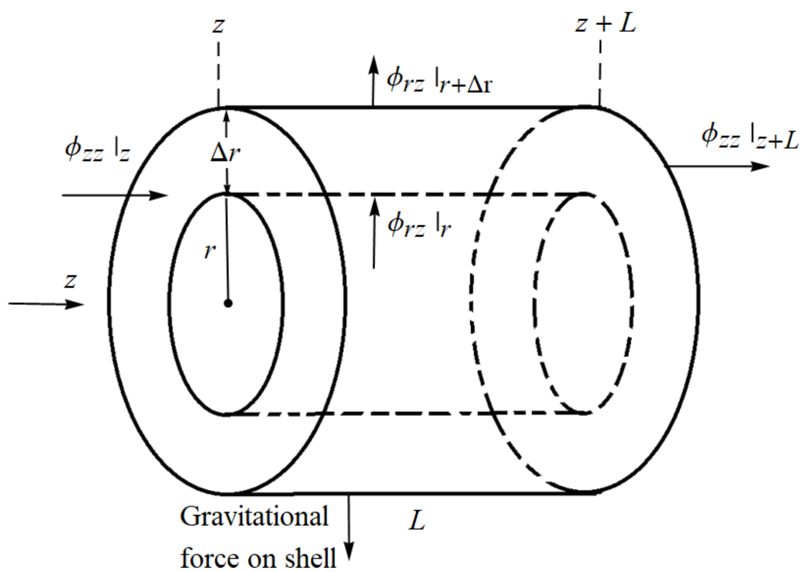

Figure 1: This is the shell over which the momentum balance is made for flow in a horizontally oriented annulus.

Rate of z-momentum into the shell at z:	$(2\pi r\Delta r)\phi_{zz}\|_z$
Rate of z-momentum out of the shell at $z+L$:	$(2\pi r\Delta r)\phi_{zz}\|_{z+L}$
Rate of z-momentum into the shell at r:	$(2\pi rL)\phi_{rz}\|_r$
Rate of z-momentum out of the shell at $r+\Delta r$:	$[2\pi(r+\Delta r)L]\phi_{rz}\|_{r+\Delta r}$
Component of gravitational force on the shell in z-direction:	0

If we assume steady flow, then the momentum balance is

$$\text{Rate of momentum in} - \text{Rate of momentum out} + \text{Force of gravity} = \mathbf{0}.$$

Considering only the z-component, we have

$$(2\pi r\Delta r)\phi_{zz}|_z - (2\pi r\Delta r)\phi_{zz}|_{z+L} + (2\pi rL)\phi_{rz}|_r - [2\pi(r+\Delta r)L]\phi_{rz}|_{r+\Delta r} = 0.$$

Factor the left side.

$$-2\pi r\Delta r(\phi_{zz}|_{z+L} - \phi_{zz}|_z) - 2\pi L[(r+\Delta r)\phi_{rz}|_{r+\Delta r} - r\phi_{rz}|_r] = 0$$

Divide both sides by $2\pi \Delta r L$.

$$-r\frac{\phi_{zz}|_{z+L} - \phi_{zz}|_z}{L} - \frac{(r+\Delta r)\phi_{rz}|_{r+\Delta r} - r\phi_{rz}|_r}{\Delta r} = 0$$

Take the limit as $\Delta r \to 0$.

$$-r\frac{\phi_{zz}|_{z+L} - \phi_{zz}|_z}{L} - \lim_{\Delta r \to 0}\frac{(r+\Delta r)\phi_{rz}|_{r+\Delta r} - r\phi_{rz}|_r}{\Delta r} = 0$$

The second term is the definition of the first derivative of $r\phi_{rz}$.

$$-r\frac{\phi_{zz}|_{z+L} - \phi_{zz}|_z}{L} - \frac{d}{dr}(r\phi_{rz}) = 0$$

Now substitute the expressions for ϕ_{rz} and ϕ_{zz}.

$$\begin{aligned}\phi_{rz} &= \tau_{rz} + \cancel{\rho v_r v_z} = \tau_{rz} \\ \phi_{zz} &= p\delta_{zz} + \cancel{\tau_{zz}} + \rho v_z v_z = p + \rho v_z^2\end{aligned}$$

Since p and v_z do not depend on z, the p and ρv_z^2 terms cancel.

$$-r\frac{\cancel{p|_{z+L}} + \cancel{\rho v_z^2|_{z+L}} - \cancel{p|_z} - \cancel{\rho v_z^2|_z}}{L} - \frac{d}{dr}(r\tau_{rz}) = 0$$

So we have

$$\frac{d}{dr}(r\tau_{rz}) = 0.$$

From Newton's law of viscosity we know that $\tau_{rz} = -\mu(dv_z/dr)$, so

$$\frac{d}{dr}\left(-\mu r\frac{dv_z}{dr}\right) = 0.$$

Integrate both sides of the differential equation with respect to r.

$$-\mu r\frac{dv_z}{dr} = C_1$$

Divide both sides by $-\mu r$.

$$\frac{dv_z}{dr} = -\frac{C_1}{\mu r}$$

Integrate both sides of the differential equation with respect to r once more.

$$v_z(r) = -\frac{C_1}{\mu}\ln r + C_2$$

Apply the boundary conditions now to determine C_1 and C_2.

$$\begin{aligned}v_z(\kappa R) &= -\frac{C_1}{\mu}\ln(\kappa R) + C_2 = v_0 \\ v_z(R) &= -\frac{C_1}{\mu}\ln R + C_2 = 0\end{aligned}$$

Solving the system of equations, we get

$$C_1 = -\frac{\mu}{\ln \kappa} v_0$$
$$C_2 = -\frac{\ln R}{\ln \kappa} v_0.$$

Armed with the constants of integration, the velocity distribution is known.

$$\begin{aligned} v_z(r) &= \frac{1}{\ln \kappa} v_0 \ln r - \frac{\ln R}{\ln \kappa} v_0 \\ &= \frac{v_0}{\ln \kappa} (\ln r - \ln R) \\ &= \frac{v_0}{\ln \kappa} \ln \frac{r}{R} \end{aligned}$$

Therefore,

$$\frac{v_z}{v_0} = \frac{\ln(r/R)}{\ln \kappa}.$$

Part (b)

The mass flow rate w is, assuming constant density ρ,

$$w = \frac{dm}{dt} = \frac{d(\rho V)}{dt} = \rho \frac{dV}{dt}.$$

The volumetric flow rate dV/dt is average velocity times cross-sectional area.

$$w = \rho \langle v_z \rangle A$$

The average velocity is obtained by integrating the velocity over the area the fluid is flowing through and then dividing by that area.

$$\begin{aligned} &= \rho \left(\frac{1}{A} \int v_z \, dA \right) A \\ &= \rho \int_{\kappa R}^{R} v_z \, (2\pi r \, dr) \\ &= 2\pi\rho \int_{\kappa R}^{R} r v_z \, dr \\ &= 2\pi\rho \int_{\kappa R}^{R} \frac{v_0}{\ln \kappa} r \ln \frac{r}{R} \, dr \end{aligned}$$

Make a substitution to solve the integral.

$$u = \frac{r}{R} \quad \rightarrow \quad r = Ru$$
$$du = \frac{dr}{R} \quad \rightarrow \quad R \, du = dr$$

The mass flow rate becomes

$$\begin{aligned} w &= 2\pi\rho \int_{\kappa}^{1} \frac{v_0}{\ln \kappa} (Ru) \ln u (R \, du) \\ &= \frac{2\pi\rho v_0 R^2}{\ln \kappa} \int_{\kappa}^{1} u \ln u \, du \\ &= \frac{2\pi\rho v_0 R^2}{\ln \kappa} \left(-\frac{u^2}{4} + \frac{u^2}{2} \ln u \right) \Bigg|_{\kappa}^{1} \end{aligned}$$

$$\begin{aligned} w &= \frac{2\pi\rho v_0 R^2}{\ln\kappa}\left(-\frac{1^2}{4} + \cancel{\frac{1^2}{2}\ln 1} + \frac{\kappa^2}{4} + \frac{\kappa^2}{2}\ln\kappa\right) \\ &= \frac{2\pi\rho v_0 R^2}{\ln\kappa}\left(\frac{\kappa^2 - 1}{4} + \frac{\kappa^2}{2}\ln\kappa\right) \\ &= \frac{2\pi\rho v_0 R^2}{4}\left(\frac{\kappa^2 - 1}{\ln\kappa} + 2\kappa^2\right) \\ &= \frac{\pi\rho v_0 R^2}{2}\left(\frac{1 - \kappa^2}{-\ln\kappa} + 2\kappa^2\right) \end{aligned}$$

Therefore,

$$w = \frac{\pi R^2 v_0 \rho}{2}\left[\frac{(1-\kappa^2)}{\ln(1/\kappa)} - 2\kappa^2\right].$$

Part (c)

The viscous stress τ_{rz} physically represents the force in the z-direction on a unit area perpendicular to the r-direction. By evaluating τ_{rz} at $r = \kappa R$ and multiplying it by the surface area of the inner cylinder, we obtain the viscous force acting on it over its length. The final point to note is that because the fluid is acting from a larger radius r on the inner cylinder, which has a smaller radius κR, we place a minus sign in front of τ_{rz}.

$$F_z = -\tau_{rz}\big|_{r=\kappa R} \cdot 2\pi(\kappa R)L$$

From Newton's law of viscosity, we have

$$\begin{aligned} \tau_{rz} &= -\mu\frac{dv_z}{dr} \\ &= -\mu\frac{d}{dr}\left(\frac{v_0}{\ln\kappa}\ln\frac{r}{R}\right) \\ &= -\mu\frac{v_0}{\ln\kappa}\frac{\cancel{R}}{r}\cdot\frac{1}{\cancel{R}} \\ &= -\frac{\mu v_0}{\ln\kappa}\cdot\frac{1}{r}. \end{aligned}$$

Now we can find the force.

$$\begin{aligned} F_z &= -\left(-\frac{\mu v_0}{\ln\kappa}\cdot\frac{1}{r}\right)\bigg|_{r=\kappa R} \cdot 2\pi(\kappa R)L \\ &= \frac{2\pi\cancel{\kappa R}L\mu v_0}{\ln\kappa}\cdot\frac{1}{\cancel{\kappa R}} \\ &= \frac{2\pi L\mu v_0}{\ln\kappa} \\ &= -\frac{2\pi L\mu v_0}{-\ln\kappa} \end{aligned}$$

Therefore,

$$F_z = -\frac{2\pi L\mu v_0}{\ln(1/\kappa)}.$$

The minus sign here makes sense because the viscous (frictional) force of the fluid opposes the motion of the cylinder, which is moving in the positive z-direction.

Part (d)

To make the annulus a plane slit, we let the radius of the inner cylinder tend towards the radius of the outer cylinder; that is, κ is very slightly less than 1.

$$\kappa = 1 - \varepsilon, \quad \text{where } 0 < \varepsilon \ll 1$$

Substitute this into the formula for F_z.

$$\begin{aligned} F_z &= \frac{2\pi L\mu v_0}{\ln\kappa} \\ &= \frac{2\pi L\mu v_0}{\ln(1-\varepsilon)} \end{aligned}$$

The Taylor series expansion for $\ln(1-\varepsilon)$ is as follows.

$$\ln(1-\varepsilon) = -\varepsilon - \frac{\varepsilon^2}{2} - \frac{\varepsilon^3}{3} - \cdots$$

Substitute this formula into F_z.

$$\begin{aligned} F_z &= \frac{2\pi L\mu v_0}{-\varepsilon - \frac{\varepsilon^2}{2} - \frac{\varepsilon^3}{3} - \cdots} \\ &= -\frac{2\pi L\mu v_0}{\varepsilon} \frac{1}{1 + \frac{\varepsilon}{2} + \frac{\varepsilon^2}{3} + \cdots} \end{aligned}$$

Use long division to obtain a series for the fraction.

$$\begin{array}{rl} & 1 - \frac{\varepsilon}{2} - \frac{\varepsilon^2}{12} - \cdots \\ 1 + \frac{\varepsilon}{2} + \frac{\varepsilon^2}{3} + \cdots \Big) & \overline{1 + 0\varepsilon + 0\varepsilon^2} \\ (-) & 1 + \frac{\varepsilon}{2} + \frac{\varepsilon^2}{3} + \cdots \\ & \overline{-\frac{\varepsilon}{2} - \frac{\varepsilon^2}{3} - \cdots} \\ (-) & -\frac{\varepsilon}{2} - \frac{\varepsilon^2}{4} - 2\varepsilon^5 - \cdots \\ & \overline{-\frac{\varepsilon^2}{12} - \cdots} \end{array}$$

Therefore,

$$\boxed{F_z = \frac{-2\pi L\mu v_0}{\varepsilon}\left(1 - \frac{\varepsilon}{2} - \frac{\varepsilon^2}{12} - \cdots\right).}$$

The viscous stress τ_{xz} for laminar slit flow with a wall moving at speed v_0 was obtained in Problem 2B.4.

$$\tau_{xz} = \left(\frac{\mathscr{P}_0 - \mathscr{P}_L}{L}\right)x - \frac{\mu v_0}{2B}$$

If we orient the slit horizontally so that pressure and gravity do not influence the fluid flow as is the case in this problem, then τ_{xz} simplifies.

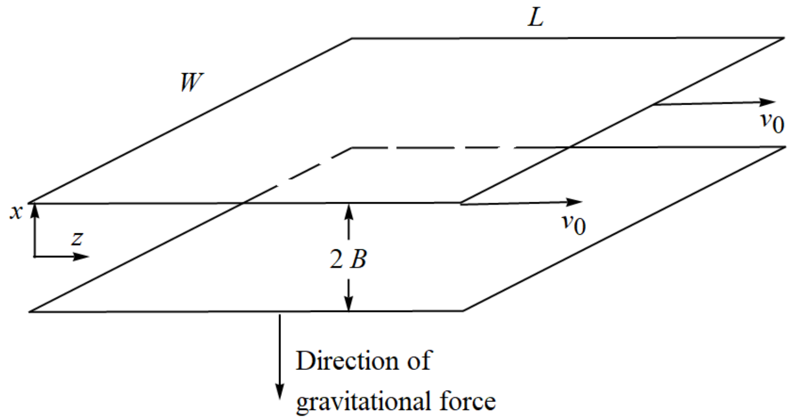

Figure 2: This is Couette flow in a horizontal slit with one moving wall at $x = B$.

$$\tau_{xz} = -\frac{\mu v_0}{2B}$$

The viscous force acting on the moving wall is $F_z = +\tau_{xz}|_{x=B} \cdot WL$. There is no minus sign here because the fluid acting on the wall has a lower x-coordinate than the moving wall at $x = B$.

$$\begin{aligned} F_z &= \left(-\frac{\mu v_0}{2B}\right)\Big|_{x=B} \cdot WL \\ &= -\frac{WL\mu v_0}{2B} \end{aligned}$$

Comparing this formula with the boxed result, we see that they are equivalent if $\varepsilon = 2B$, the slit width, and $W = 2\pi$. W should be a distance, but it's only an angle; thus, the fraction in the boxed result is not truly a plane slit formula. The remaining series in parentheses then must be a curvature correction.

$$F_z = \underbrace{-\frac{2\pi L\mu v_0}{\varepsilon}}_{\text{"plane slit" formula}} \underbrace{\left(1 - \frac{\varepsilon}{2} - \frac{\varepsilon^2}{12} - \cdots\right)}_{\text{curvature correction}}$$