Problem 2B.8

Analysis of a capillary flowmeter (see Fig. 2B.8).

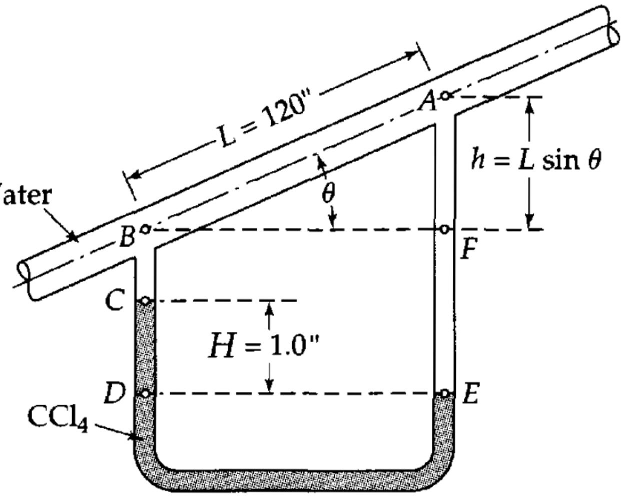

Fig. 2B.8 A capillary flow meter.

Determine the rate of flow (in lb_m/hr) through the capillary flow meter shown in the figure. The fluid flowing in the inclined tube is water at 20°C, and the manometer fluid is carbon tetrachloride (CCl_4) with density 1.594 g/cm^3. The capillary diameter is 0.010 in. *Note:* Measurements of H and L are sufficient to calculate the flow rate; θ need not be measured. Why?

Solution

For the capillary flow meter, choose a cylindrical coordinate system with the positive z-direction pointing in the direction of the flow. The fluid velocity is assumed to vary as a function of radius r.

$$v_z = v_z(r)$$

As a result, only ϕ_{rz} (the z-momentum in the positive r-direction) and ϕ_{zz} (the z-momentum in the positive z-direction) contribute to the momentum balance. Figure 1 on the next page shows the shell the momentum balance is made over.

Rate of z-momentum into the shell at $z = 0$:	$(2\pi r\Delta r)\phi_{zz}\|_{z=0}$
Rate of z-momentum out of the shell at $z = L$:	$(2\pi r\Delta r)\phi_{zz}\|_{z=L}$
Rate of z-momentum into the shell at r:	$(2\pi rL)\phi_{rz}\|_r$
Rate of z-momentum out of the shell at $r + \Delta r$:	$[2\pi(r+\Delta r)L]\phi_{rz}\|_{r+\Delta r}$
Component of gravitational force on the shell in z-direction:	$(2\pi r\Delta rL)\rho g\sin\theta$

If we assume steady flow, then the momentum balance is

$$\text{Rate of momentum in} - \text{Rate of momentum out} + \text{Force of gravity} = \mathbf{0}.$$

Considering only the z-component, we have

$$(2\pi r\Delta r)\phi_{zz}|_{z=0} - (2\pi r\Delta r)\phi_{zz}|_{z=L} + (2\pi rL)\phi_{rz}|_r - [2\pi(r+\Delta r)L]\phi_{rz}|_{r+\Delta r} + (2\pi r\Delta rL)\rho g\sin\theta = 0.$$

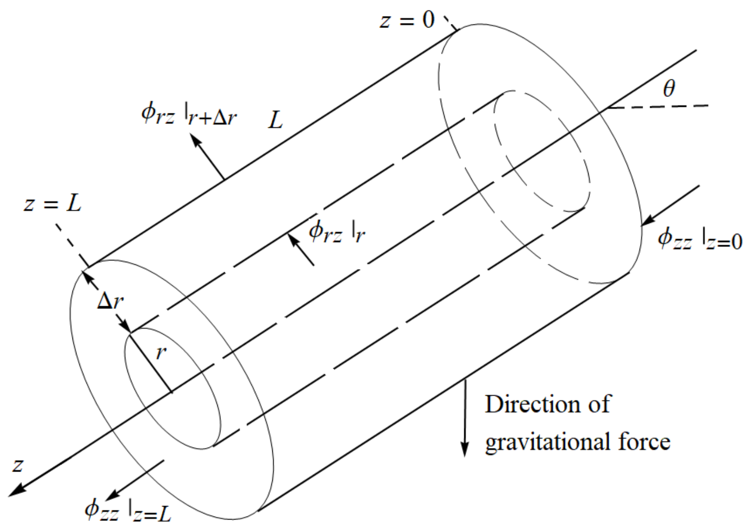

Figure 1: This is the shell over which the momentum balance is made for fluid going through the capillary flow meter.

Factor the left side.

$$-2\pi r\Delta r(\phi_{zz}|_{z=L} - \phi_{zz}|_{z=0}) - 2\pi L[(r+\Delta r)\phi_{rz}|_{r+\Delta r} - r\phi_{rz}|_r] + 2\pi r\Delta r L\rho g\sin\theta = 0$$

Divide both sides by $2\pi\Delta r L$.

$$-r\frac{\phi_{zz}|_{z=L} - \phi_{zz}|_{z=0}}{L} - \frac{(r+\Delta r)\phi_{rz}|_{r+\Delta r} - r\phi_{rz}|_r}{\Delta r} + \rho g r\sin\theta = 0$$

Take the limit as $\Delta r \to 0$.

$$-r\frac{\phi_{zz}|_{z=L} - \phi_{zz}|_{z=0}}{L} - \lim_{\Delta r\to 0}\frac{(r+\Delta r)\phi_{rz}|_{r+\Delta r} - r\phi_{rz}|_r}{\Delta r} + \rho g r\sin\theta = 0$$

The second term is the definition of the first derivative of $r\phi_{rz}$.

$$-r\frac{\phi_{zz}|_{z=L} - \phi_{zz}|_{z=0}}{L} - \frac{d}{dr}(r\phi_{rz}) + \rho g r\sin\theta = 0$$

Now substitute the expressions for ϕ_{rz} and ϕ_{zz}.

$$\begin{aligned}\phi_{rz} &= \tau_{rz} + \cancel{\rho v_r v_z} = \tau_{rz}\\ \phi_{zz} &= p\delta_{zz} + \cancel{\tau_{zz}} + \rho v_z v_z = p + \rho v_z^2\end{aligned}$$

Since v_z does not depend on z, the ρv_z^2 terms cancel.

$$-r\frac{p|_{z=L} + \cancel{\rho v_z^2|_{z=L}} - p|_{z=0} - \cancel{\rho v_z^2|_{z=0}}}{L} - \frac{d}{dr}(r\tau_{rz}) + \rho g r\sin\theta = 0$$

Make it so that $\rho g r\sin\theta$ is in the fraction.

$$-r\frac{p|_{z=L} - p|_{z=0} - \rho g L\sin\theta}{L} - \frac{d}{dr}(r\tau_{rz}) = 0$$

From the schematic in Fig. 2B.8, we see that $h = L\sin\theta$. It is here where θ disappears from the equation; hence, it does not need to be measured.

$$-r\frac{p|_{z=L} - p|_{z=0} - \rho g h}{L} - \frac{d}{dr}(r\tau_{rz}) = 0$$

So we have

$$\frac{d}{dr}(r\tau_{rz}) = \frac{\rho g h + (p|_{z=0} - p|_{z=L})}{L} r.$$

It is thanks to the manometer underneath the capillary that the principles of fluid statics can be applied to compute the quantity in the numerator. Let

$$\begin{aligned} \rho &= \text{the density of water} \\ \rho_C &= \text{the density of CCl}_4 \\ H' &= \text{the distance between points } B \text{ and } C \text{ in Fig. 2B.8.} \end{aligned}$$

D and E are at the same height in the CCl_4; thus, the pressures at these levels must be equal.

$$\underbrace{p|_{z=L} + \rho g H' + \rho_C g H}_{\text{pressure at } D} = \underbrace{p|_{z=0} + \rho g h + \rho g H' + \rho g H}_{\text{pressure at } E}$$

The $\rho g H'$ terms cancel, so the distance between points B and C is not needed. Solve this equation for the quantity in the numerator

$$\rho g h + p|_{z=0} - p|_{z=L} = (\rho_C - \rho) g H$$

and substitute it into the differential equation.

$$\frac{d}{dr}(r\tau_{rz}) = \frac{(\rho_C - \rho) g H}{L} r$$

From Newton's law of viscosity we know that $\tau_{rz} = -\mu(dv_z/dr)$, so

$$\frac{d}{dr}\left(-\mu r \frac{dv_z}{dr}\right) = \frac{(\rho_C - \rho) g H}{L} r.$$

We thus have a differential equation for the velocity distribution in the capillary. The boundary conditions for it are obtained from the assumptions that the velocity is maximum furthest from the wall (at $r = 0$) and that no slipping occurs between the fluid and the wall (at $r = R$).

$$\begin{aligned} \text{B.C. 1}: &\quad \frac{dv_z}{dr} = 0 \quad \text{when} \quad r = 0 \\ \text{B.C. 2}: &\quad v_z = 0 \quad \text{when} \quad r = R \end{aligned}$$

Integrate both sides of the differential equation with respect to r.

$$-\mu r \frac{dv_z}{dr} = \frac{(\rho_C - \rho) g H}{2L} r^2 + C_1$$

Apply the first boundary condition now to determine C_1.

$$-\mu(0)\left.\frac{dv_z}{dr}\right|_{r=0} = \frac{(\rho_C - \rho) g H}{2L}(0)^2 + C_1 \quad \rightarrow \quad 0 = C_1$$

Divide both sides by $-\mu r$.

$$\frac{dv_z}{dr} = -\frac{(\rho_C - \rho)gH}{2\mu L} r$$

Integrate both sides of the differential equation with respect to r once more.

$$v_z(r) = -\frac{(\rho_C - \rho)gH}{4\mu L} r^2 + C_2$$

Apply the second boundary condition now to determine C_2.

$$v_z(R) = -\frac{(\rho_C - \rho)gH}{4\mu L} R^2 + C_2 = 0 \quad \rightarrow \quad C_2 = \frac{(\rho_C - \rho)gH}{4\mu L} R^2$$

With the constants of integration in hand, the velocity distribution is known.

$$\begin{aligned} v_z(r) &= -\frac{(\rho_C - \rho)gH}{4\mu L} r^2 + \frac{(\rho_C - \rho)gH}{4\mu L} R^2 \\ &= \frac{(\rho_C - \rho)gH}{4\mu L} (R^2 - r^2) \end{aligned}$$

This result can be used to obtain the mass rate of flow w.

$$w = \frac{dm}{dt} = \frac{d(\rho V)}{dt} = \rho \frac{dV}{dt}$$

The volumetric flow dV/dt is equal to the average velocity in the capillary times its cross-sectional area.

$$w = \rho \langle v_z \rangle \cdot \pi R^2$$

The average velocity is found by integrating v_z over the area of the cross-section and then dividing by that area.

$$\begin{aligned} w &= \rho \left(\frac{1}{\cancel{\pi R^2}} \int v_z \, dA \right) \cdot \cancel{\pi R^2} \\ &= \rho \int_0^R v_z (2\pi r \, dr) \\ &= 2\pi\rho \int_0^R r v_z \, dr \\ &= 2\pi\rho \int_0^R r \frac{(\rho_C - \rho)gH}{4\mu L} (R^2 - r^2) \, dr \\ &= \frac{\pi\rho(\rho_C - \rho)gH}{2\mu L} \int_0^R (rR^2 - r^3) \, dr \\ &= \frac{\pi\rho(\rho_C - \rho)gH}{2\mu L} \left(\frac{r^2 R^2}{2} - \frac{r^4}{4} \right) \Bigg|_0^R \\ &= \frac{\pi\rho(\rho_C - \rho)gH}{2\mu L} \left(\frac{R^4}{2} - \frac{R^4}{4} \right) \\ &= \frac{\pi\rho(\rho_C - \rho)gHR^4}{8\mu L} \end{aligned}$$

Use Eq. 1.1-3, $\nu = \mu/\rho$, to write the density of water ρ in terms of the kinematic viscosity ν and viscosity μ. Also, write the radius in terms of the diameter.

$$\begin{aligned} w &= \frac{\pi\left(\rho_C - \frac{\mu}{\nu}\right) gH \left(\frac{D}{2}\right)^4}{8\nu L} \\ &= \frac{\pi\left(\rho_C\nu - \mu\right) gHD^4}{128\nu^2 L} \end{aligned}$$

Before we plug in the numbers, convert the units so that the desired units of lb_m/hr are obtained. μ and ν for water at 20°C are given on page 14 in Table 1.1-2. Other conversion factors are on page 868 and 870.

$$\begin{aligned} \rho_C &= 1.594\ \frac{\cancel{\text{g}}}{\cancel{\text{cm}^3}} \times \left(\frac{2.54\ \cancel{\text{cm}}}{1\ \text{in}}\right)^3 \times \frac{1\ \cancel{\text{kg}}}{1000\ \cancel{\text{g}}} \times \frac{2.2046\ \text{lb}_m}{1\ \cancel{\text{kg}}} \approx 0.0575863\ \frac{\text{lb}_m}{\text{in}^3} \\ \nu &= 0.010037\ \frac{\cancel{\text{cm}^2}}{\cancel{\text{s}}} \times \left(\frac{1\ \text{in}}{2.54\ \cancel{\text{cm}}}\right)^2 \times \frac{3600\ \cancel{\text{s}}}{1\ \text{hr}} \approx 5.60066\ \frac{\text{in}^2}{\text{hr}} \\ \mu &= 1.0019\ \cancel{\text{mPa}\cdot\text{s}} \times \frac{2.4191\ \frac{\text{lb}_m}{\cancel{\text{ft}}\cdot\text{hr}}}{1\ \cancel{\text{mPa}\cdot\text{s}}} \times \frac{1\ \cancel{\text{ft}}}{12\ \text{in}} \approx 0.201975\ \frac{\text{lb}_m}{\text{in}\cdot\text{hr}} \\ g &= 9.81\ \frac{\cancel{\text{m}}}{\cancel{\text{s}^2}} \times \frac{3.28\ \cancel{\text{ft}}}{1\ \cancel{\text{m}}} \times \frac{12\ \text{in}}{1\ \cancel{\text{ft}}} \times \left(\frac{3600\ \cancel{\text{s}}}{1\ \text{hr}}\right)^2 \approx 5.00414 \times 10^9\ \frac{\text{in}}{\text{hr}^2} \\ H &= 1.0\ \text{in} \\ D &= 0.010\ \text{in} \\ L &= 120\ \text{in} \end{aligned}$$

Therefore,

$$\begin{aligned} w &\approx \frac{\pi(0.0576 \cdot 5.6 - 0.202)(5\times 10^9)(1.0)(0.010)^4\ \frac{\text{lb}_m\cdot\text{in}^5}{\text{hr}^3}}{128(5.6)^2(120)\ \frac{\text{in}^5}{\text{hr}^2}} \\ &\approx 3.9 \times 10^{-5}\ \frac{\text{lb}_m}{\text{hr}}. \end{aligned}$$