Problem 2C.1

Performance of an electric dust collector (see Fig. 2C.1)[5].

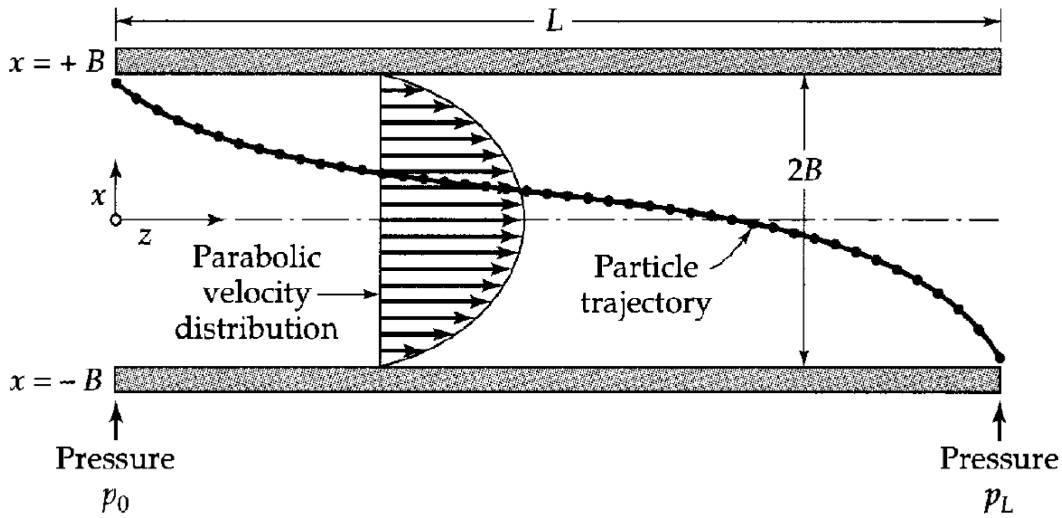

Figure 1: This is Fig. 2C.1 in the text. A possible particle trajectory. The particle that begins at $z = 0$, $x = B$ and ends at $x = -B$ may not necessarily travel the longest distance in the z direction.

(a) A dust precipitator consists of a pair of oppositely charged plates between which dust-laden gases flow. It is desired to establish a criterion for the minimum length of the precipitator in terms of the charge on the particle e, the electric field strength $\mathscr{E}$, the pressure difference $(p_0 - p_L)$, the particle mass m, and the gas viscosity μ. That is, for what length L will the smallest particle present (mass m) reach the bottom just before it has a chance to be swept out of the channel? Assume that the flow between the plates is laminar so that the velocity distribution is described by Eq. 2B.3-2. Assume also that the particle velocity in the z direction is the same as the fluid velocity in the z direction. Assume further that the Stokes drag on the sphere as well as the gravity force acting on the sphere as it is accelerated in the negative x direction can be neglected.

(b) Rework the problem neglecting acceleration in the x direction, but including the Stokes drag.

(c) Compare the usefulness of the solutions in (a) and (b), considering that stable aerosol particles have effective diameters of about 1-10 microns and densities of about 1 g/cm^3.

Answer: **(a)** $L_{\min} = [12(p_0 - p_L)^2 B^5 m / 25\mu^2 e\mathscr{E}]^{1/4}$

Solution

[5] The answer given in the first edition of this book was incorrect, as pointed out to us in 1970 by Nau Gab Lee of Seoul National University.

Part (a)

Consider a particle with mass m and charge e in the slit. If there's an electric field in the x-direction, then the particle will accelerate in that direction until it hits the lower wall. Our aim is to find out how long it takes for this to happen. Once this time is known, we will use the fluid velocity in the z-direction to determine how far it travels in the z-direction. Essentially this is a kinematics problem. A charge e in an electric field $\mathscr{E}$ experiences a force $e\mathscr{E}$. Since the charge is moving from positive x to negative x, the electric field is negative. Apply Newton's second law in the x-direction to determine the particle's acceleration in this direction.

$$\sum F_x = e(-\mathscr{E}) = ma_x$$

Acceleration is the second derivative of position with respect to time.

$$m\frac{d^2x}{dt^2} = -e\mathscr{E}$$

Divide both sides by m.

$$\frac{d^2x}{dt^2} = -\frac{e\mathscr{E}}{m}$$

Initially the particle is assumed to be at rest, so one of the initial conditions is $x'(0) = 0$. The second one is trickier because as mentioned in the problem statement, a particle that starts at $z = 0$, $x = B$ won't necessarily travel the furthest distance in the z-direction. What we will do then is assume an arbitrary initial position $x(0) = x_0$ and then in the end choose the value of x_0 that maximizes L. Integrate the differential equation with respect to t.

$$\frac{dx}{dt} = -\frac{e\mathscr{E}}{m}t + C_1$$

Apply the first initial condition to determine C_1.

$$x'(0) = -\frac{e\mathscr{E}}{m}(0) + C_1 = 0 \quad \rightarrow \quad C_1 = 0$$

Integrate the differential equation with respect to t once more.

$$x(t) = -\frac{e\mathscr{E}}{2m}t^2 + C_2$$

Apply the second initial condition to determine C_2.

$$x(0) = -\frac{e\mathscr{E}}{2m}(0)^2 + C_2 = x_0 \quad \rightarrow \quad C_2 = x_0$$

So we have

$$x(t) = x_0 - \frac{e\mathscr{E}}{2m}t^2. \tag{1}$$

Plug in $x = -B$ and solve for t to find how long it takes for the particle to go from its initial position to the lower wall at $x = -B$.

$$-B = x_0 - \frac{e\mathscr{E}}{2m}t^2 \quad \rightarrow \quad \frac{e\mathscr{E}}{2m}t^2 = x_0 + B \quad \rightarrow \quad t = \sqrt{\frac{2m}{e\mathscr{E}}(x_0 + B)}$$

Now that we know how long the particle is moving for, we can determine how far it moves in the z-direction. For a fluid in laminar flow, the formula for the velocity in a vertical slit is given by Eq. 2B.3-2 on page 63.

$$\begin{aligned} v_z(x) &= \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2}{2\mu L}\left[1 - \left(\frac{x}{B}\right)^2\right] \\ &= \frac{\mathscr{P}_0 - \mathscr{P}_L}{2\mu L}(B^2 - x^2) \end{aligned} \tag{2B.3-2}$$

Since the slit we're dealing with here is extended horizontally, gravity does not affect the motion of the fluid, so the modified pressure $\mathscr{P}$ reduces to the pressure p.

$$v_z(x) = \frac{p_0 - p_L}{2\mu L}(B^2 - x^2)$$

Substitute equation (1) for x to get v_z as a function of t.

$$v_z(t) = \frac{p_0 - p_L}{2\mu L}\left[B^2 - \left(x_0 - \frac{e\mathscr{E}}{2m}t^2\right)^2\right]$$

Expand the terms inside the square brackets.

$$v_z(t) = \frac{p_0 - p_L}{2\mu L}\left(B^2 - x_0^2 + \frac{e\mathscr{E}}{m}x_0 t^2 - \frac{e^2\mathscr{E}^2}{4m^2}t^4\right)$$

To get the distance traveled in the z-direction, integrate the velocity with respect to time. No constant of integration is necessary because the particle starts from $z = 0$.

$$z(t) = \int v_z\, dt$$

By choosing the lower limit of integration to be when the particle starts moving (at $t = 0$) and the upper limit of integration to be when the particle hits the wall at $x = -B$ and stops moving, we obtain L, the distance in the z-direction that the particle travels.

$$\begin{aligned} L &= \int_0^{\sqrt{\frac{2m}{e\mathscr{E}}(x_0+B)}} v_z\, dt \\ &= \int_0^{\sqrt{\frac{2m}{e\mathscr{E}}(x_0+B)}} \frac{p_0 - p_L}{2\mu L}\left(B^2 - x_0^2 + \frac{e\mathscr{E}}{m}x_0 t^2 - \frac{e^2\mathscr{E}^2}{4m^2}t^4\right) dt \\ &= \frac{p_0 - p_L}{2\mu L}\left[(B^2 - x_0^2)\int_0^{\sqrt{\frac{2m}{e\mathscr{E}}(x_0+B)}} dt + \frac{e\mathscr{E}}{m}x_0\int_0^{\sqrt{\frac{2m}{e\mathscr{E}}(x_0+B)}} t^2\, dt - \frac{e^2\mathscr{E}^2}{4m^2}\int_0^{\sqrt{\frac{2m}{e\mathscr{E}}(x_0+B)}} t^4\, dt\right] \\ &= \frac{p_0 - p_L}{2\mu L}\left\{(B^2 - x_0^2)\sqrt{\frac{2m}{e\mathscr{E}}(x_0+B)} + \frac{e\mathscr{E}}{3m}x_0\left[\sqrt{\frac{2m}{e\mathscr{E}}(x_0+B)}\right]^3 - \frac{e^2\mathscr{E}^2}{20m^2}\left[\sqrt{\frac{2m}{e\mathscr{E}}(x_0+B)}\right]^5\right\} \\ &= \frac{p_0 - p_L}{2\mu L}\left\{(B^2 - x_0^2) + \frac{e\mathscr{E}}{3m}x_0\left[\frac{2m}{e\mathscr{E}}(x_0+B)\right] - \frac{e^2\mathscr{E}^2}{20m^2}\left[\frac{2m}{e\mathscr{E}}(x_0+B)\right]^2\right\}\sqrt{\frac{2m}{e\mathscr{E}}(x_0+B)} \\ &= \frac{p_0 - p_L}{2\mu L}\left[(B^2 - x_0^2) + \frac{2}{3}x_0(x_0+B) - \frac{1}{5}(x_0+B)^2\right]\sqrt{\frac{2m}{e\mathscr{E}}(x_0+B)} \end{aligned}$$

$$\begin{aligned}
L &= \frac{p_0 - p_L}{2\mu L}\left(\frac{4}{5}B^2 + \frac{4}{15}Bx_0 - \frac{8}{15}x_0^2\right)\sqrt{\frac{2m}{e\mathscr{E}}(x_0 + B)} \\
&= \frac{p_0 - p_L}{2\mu L}\cdot\frac{4}{15}(3B - 2x_0)(x_0 + B)\sqrt{\frac{2m}{e\mathscr{E}}(x_0 + B)} \\
&= \frac{p_0 - p_L}{\mu L}\cdot\frac{2}{15}\sqrt{\frac{2m}{e\mathscr{E}}}(3B - 2x_0)(x_0 + B)^{3/2}
\end{aligned}$$

Multiply both sides of the equation by L.

$$L^2 = \frac{p_0 - p_L}{\mu}\cdot\frac{2}{15}\sqrt{\frac{2m}{e\mathscr{E}}}(3B - 2x_0)(x_0 + B)^{3/2} \tag{2}$$

Now we're ready to determine the initial position x_0 in the slit that maximizes L. Take the derivative of L^2 with respect to x_0 and set the result equal to 0.

$$\frac{d}{dx_0}(L^2) = \frac{p_0 - p_L}{\mu}\cdot\frac{2}{15}\sqrt{\frac{2m}{e\mathscr{E}}}\cdot\frac{5}{2}(B - 2x_0)\sqrt{B + x_0} = 0$$

This is only satisfied if

$$\begin{aligned}
B - 2x_0 = 0 \quad &\rightarrow \quad x_0 = \frac{B}{2} \\
B + x_0 = 0 \quad &\rightarrow \quad x_0 = -B.
\end{aligned}$$

The maximum value of L is obtained if the particle's initial position is $x_0 = B/2$. Now plug this into equation (2) to determine the maximum value of L^2.

$$\begin{aligned}
L^2 &= \frac{p_0 - p_L}{\mu}\cdot\frac{2}{15}\sqrt{\frac{2m}{e\mathscr{E}}}\left(3B - 2\frac{B}{2}\right)\left(\frac{B}{2} + B\right)^{3/2} \\
&= \frac{p_0 - p_L}{\mu}\cdot\frac{2}{15}\sqrt{\frac{2m}{e\mathscr{E}}}(2B)\left(\frac{3B}{2}\right)^{3/2} \\
&= \frac{p_0 - p_L}{\mu}\sqrt{\frac{m}{e\mathscr{E}}}\sqrt{\frac{12}{25}}\sqrt{B^5}
\end{aligned}$$

Take the square root of both sides to solve for L.

$$L = \sqrt{\frac{p_0 - p_L}{\mu}\sqrt{\frac{m}{e\mathscr{E}}}\sqrt{\frac{12}{25}}\sqrt{B^5}}$$

Use the fourth root to simplify the answer.

$$L = \sqrt[4]{\left(\frac{p_0 - p_L}{\mu}\right)^2\cdot\frac{m}{e\mathscr{E}}\cdot\frac{12}{25}\cdot B^5}$$

Therefore,

$$L_{\text{min}} = \sqrt[4]{\frac{12(p_0 - p_L)^2 B^5 m}{25\mu^2 e\mathscr{E}}}.$$

The subscript "min" refers to the fact that this is the length for the smallest mass.

Part (b)

Here the acceleration in the x-direction will be neglected (i.e. a_x set equal to 0) and the Stokes drag ($F_k = 6\pi\mu R v_\infty$) will be included in the sum of the forces. Since $a_x = 0$, the particle will be moving at terminal velocity from its initial position $x = x_0$ to the lower wall $x = -B$. The Stokes drag acts in the direction opposing the particle's motion, so it will be in the positive x-direction. As in part (a), the electric field is responsible for pushing the particle down. Consider Newton's second law in the x-direction.

$$\sum F_x = e(-\mathscr{E}) + 6\pi\mu R v_\infty = m a_x = 0$$

Solve for v_∞, the velocity that the fluid is approaching the particle from below.

$$v_\infty = \frac{e\mathscr{E}}{6\pi\mu R}$$

The terminal velocity is the negative of this, as the particle is moving down.

$$v_t = -\frac{e\mathscr{E}}{6\pi\mu R}$$

Velocity is the derivative of position with respect to time.

$$\frac{dx}{dt} = -\frac{e\mathscr{E}}{6\pi\mu R}$$

As mentioned in the problem statement, a particle that starts at $z = 0$, $x = B$ won't necessarily travel the furthest distance in the z-direction. What we will do then is assume an arbitrary initial position $x(0) = x_0$ and then in the end choose the value of x_0 that maximizes L. Integrate the differential equation with respect to t.

$$x(t) = -\frac{e\mathscr{E}}{6\pi\mu R}t + C_3$$

Apply the initial condition to determine C_3.

$$x(0) = -\frac{e\mathscr{E}}{6\pi\mu R}(0) + C_3 = x_0 \quad \rightarrow \quad C_3 = x_0$$

So we have

$$x(t) = x_0 - \frac{e\mathscr{E}}{6\pi\mu R}t. \tag{3}$$

Plug in $x = -B$ and solve for t to find how long it takes for the particle to go from its initial position to the lower wall at $x = -B$.

$$-B = x_0 - \frac{e\mathscr{E}}{6\pi\mu R}t \quad \rightarrow \quad \frac{e\mathscr{E}}{6\pi\mu R}t = x_0 + B \quad \rightarrow \quad t = \frac{6\pi\mu R}{e\mathscr{E}}(x_0 + B)$$

Now that we know how long the particle is moving for, we can determine how far it moves in the z-direction. As explained in part (a), the velocity for the flow in a horizontally extended slit is

$$v_z(x) = \frac{p_0 - p_L}{2\mu L}(B^2 - x^2).$$

Substitute equation (3) for x to get v_z as a function of t.

$$v_z(t) = \frac{p_0 - p_L}{2\mu L}\left[B^2 - \left(x_0 - \frac{e\mathscr{E}}{6\pi\mu R}t\right)^2\right]$$

Expand the terms inside the square brackets.

$$v_z(t) = \frac{p_0 - p_L}{2\mu L}\left(B^2 - x_0^2 + \frac{e\mathscr{E}}{3\pi\mu R}x_0 t - \frac{e^2\mathscr{E}^2}{36\pi^2\mu^2R^2}t^2\right)$$

To get the distance traveled in the z-direction, integrate the velocity with respect to time. No constant of integration is necessary because the particle starts from $z = 0$.

$$z(t) = \int v_z\,dt$$

By choosing the lower limit of integration to be when the particle starts moving (at $t = 0$) and the upper limit of integration to be when the particle hits the wall at $x = -B$ and stops moving, we obtain L, the distance in the z-direction that the particle travels.

$$\begin{aligned}
L &= \int_0^{\frac{6\pi\mu R}{e\mathscr{E}}(x_0+B)} v_z\,dt \\
&= \int_0^{\frac{6\pi\mu R}{e\mathscr{E}}(x_0+B)} \frac{p_0 - p_L}{2\mu L}\left(B^2 - x_0^2 + \frac{e\mathscr{E}}{3\pi\mu R}x_0 t - \frac{e^2\mathscr{E}^2}{36\pi^2\mu^2R^2}t^2\right)dt \\
&= \frac{p_0 - p_L}{2\mu L}\left[(B^2 - x_0^2)\int_0^{\frac{6\pi\mu R}{e\mathscr{E}}(x_0+B)} dt + \frac{e\mathscr{E}}{3\pi\mu R}x_0\int_0^{\frac{6\pi\mu R}{e\mathscr{E}}(x_0+B)} t\,dt - \frac{e^2\mathscr{E}^2}{36\pi^2\mu^2R^2}\int_0^{\frac{6\pi\mu R}{e\mathscr{E}}(x_0+B)} t^2\,dt\right] \\
&= \frac{p_0 - p_L}{2\mu L}\left\{(B^2 - x_0^2)\left[\frac{6\pi\mu R}{e\mathscr{E}}(x_0+B)\right] + \frac{e\mathscr{E}}{6\pi\mu R}x_0\left[\frac{6\pi\mu R}{e\mathscr{E}}(x_0+B)\right]^2 - \frac{e^2\mathscr{E}^2}{108\pi^2\mu^2R^2}\left[\frac{6\pi\mu R}{e\mathscr{E}}(x_0+B)\right]^3\right\} \\
&= \frac{p_0 - p_L}{2\mu L}\left[\frac{6\pi\mu R}{e\mathscr{E}}(B^2 - x_0^2)(x_0+B) + \frac{6\pi\mu R}{e\mathscr{E}}x_0(x_0+B)^2 - \frac{2\pi\mu R}{e\mathscr{E}}(x_0+B)^3\right] \\
&= \frac{p_0 - p_L}{2\mu L}\cdot\frac{6\pi\mu R}{e\mathscr{E}}\left[(B^2 - x_0^2) + x_0(x_0+B) - \frac{1}{3}(x_0+B)^2\right](x_0+B) \\
&= \frac{p_0 - p_L}{L}\cdot\frac{3\pi R}{e\mathscr{E}}\cdot\frac{1}{3}(2B - x_0)(x_0+B)(x_0+B) \\
&= \frac{p_0 - p_L}{L}\cdot\frac{\pi R}{e\mathscr{E}}(2B - x_0)(x_0+B)^2
\end{aligned}$$

Multiply both sides by L.

$$L^2 = \frac{\pi(p_0 - p_L)R}{e\mathscr{E}}(2B - x_0)(x_0+B)^2 \tag{4}$$

Now we're ready to determine the initial position x_0 in the slit that maximizes L. Take the derivative of L^2 with respect to x_0 and set the result equal to 0.

$$\frac{d}{dx_0}(L^2) = \frac{\pi(p_0 - p_L)R}{e\mathscr{E}}\cdot 3(B - x_0)(B + x_0) = 0$$

This is only satisfied if

$$\begin{aligned}
B - x_0 = 0 \quad &\rightarrow \quad x_0 = B \\
B + x_0 = 0 \quad &\rightarrow \quad x_0 = -B.
\end{aligned}$$

The maximum value of L is obtained if the particle's initial position is $x_0 = B$. Now plug this into equation (4) to determine the maximum value of L^2.

$$\begin{aligned} L^2 &= \frac{\pi(p_0 - p_L)R}{e\mathscr{E}}(2B - B)(B + B)^2 \\ &= \frac{\pi(p_0 - p_L)R}{e\mathscr{E}} 4B^3 \\ &= \frac{4\pi(p_0 - p_L)B^3 R}{e\mathscr{E}} \end{aligned}$$

Therefore,

$$L_{\min} = \sqrt{\frac{4\pi(p_0 - p_L)B^3 R}{e\mathscr{E}}}.$$

Interestingly $L_{\min}$ is independent of the particle's mass and the fluid's viscosity. Also, releasing the particle at $x = B$ does lead to the maximum value of L in this case because the acceleration in the x-direction was neglected.

Part (c)

With the diameter and density of an aerosol particle, one can compute its mass by

$$\begin{aligned} m &= \rho \cdot V \\ &= \rho \cdot \frac{4}{3}\pi R^3 \end{aligned}$$

In SI units the mass is exceedingly small: a 10 micron diameter particle has a mass of about 5×10^{-13} kg. By making the assumption in part (a) that the Stokes drag can be neglected, one is effectively saying that it is negligible compared to the inertial force ma_x in Newton's second law.

$$\sum F_x = e(-\mathscr{E}) + \cancel{6\pi\mu R v_\infty} = ma_x$$

Given how small the mass is, this is not justified, and I would deem the result in part (a) to be useless. On the other hand, neglecting the acceleration in the x-direction is justified because of the small mass, which is what we do in part (b).

$$\sum F_x = e(-\mathscr{E}) + 6\pi\mu R v_\infty = \cancel{ma_x}$$

Because of the really small diameter of aerosol particles (10^{-5} m to 10^{-6} m), the Reynolds number,

$$\text{Re} = \frac{Dv_\infty \rho}{\mu},$$

will be less than 0.1 for reasonable fluids and velocities. Stokes's law will be valid as a result, and the answer in part (b) should give an accurate value for $L_{\min}$. Also, on page 61 it says Stokes's law is used in the study of the motion of aerosol particles. That tells us something about its utility.