Problem 2C.3

Velocity distribution in a tube. You have received a manuscript to referee for a technical journal. The paper deals with heat transfer in tube flow. The authors state that, because they are concerned with nonisothermal flow, they must have a "general" expression for the velocity distribution, one that can be used even when the viscosity of the fluid is a function of temperature (and hence position). The authors state that a "general expression for the velocity distribution for flow in a tube" is

$$\frac{v_z}{\langle v_z \rangle} = \frac{\displaystyle\int_y^1 (\bar{y}/\mu)\,d\bar{y}}{\displaystyle\int_0^1 (\bar{y}^3/\mu)\,d\bar{y}} \tag{2C.3-1}$$

in which $y = r/R$. The authors give no derivation, nor do they give a literature citation. As the referee you feel obliged to derive the formula and list any restrictions implied.

Solution

We assume that the fluid flows only in the z-direction and that its velocity varies with radius r.

$$v_z = v_z(r)$$

As a result, only ϕ_{rz} (the z-momentum in the positive r-direction) and ϕ_{zz} (the z-momentum in the positive z-direction) contribute to the momentum balance. The pressure is assumed to vary with height z.

$$p = p(z)$$

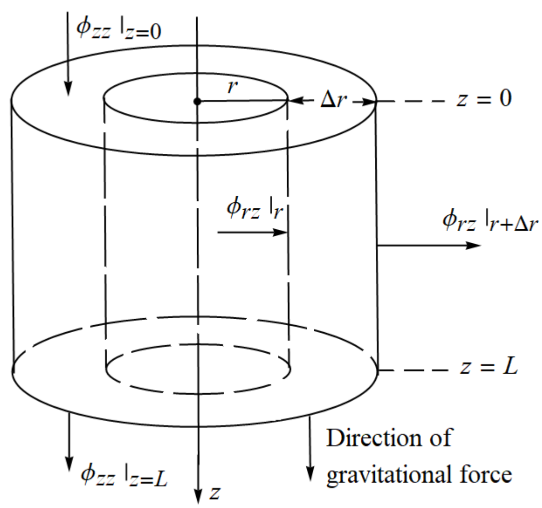

Figure 1: This is the shell over which the momentum balance is made for flow through a cylindrical tube oriented vertically.

Rate of z-momentum into the shell at $z = 0$:	$(2\pi r\Delta r)\phi_{zz}\|_{z=0}$
Rate of z-momentum out of the shell at $z = L$:	$(2\pi r\Delta r)\phi_{zz}\|_{z=L}$
Rate of z-momentum into the shell at r:	$(2\pi rL)\phi_{rz}\|_{r}$
Rate of z-momentum out of the shell at $r + \Delta r$:	$[2\pi(r+\Delta r)L]\phi_{rz}\|_{r+\Delta r}$
Component of gravitational force on the shell in z-direction:	$(2\pi r\Delta rL)\rho g$

If we assume steady flow, then the momentum balance is

$$\text{Rate of momentum in} - \text{Rate of momentum out} + \text{Force of gravity} = \mathbf{0}.$$

Considering only the z-component, we have

$$(2\pi r\Delta r)\phi_{zz}|_{z=0} - (2\pi r\Delta r)\phi_{zz}|_{z=L} + (2\pi rL)\phi_{rz}|_{r} - [2\pi(r+\Delta r)L]\phi_{rz}|_{r+\Delta r} + (2\pi r\Delta rL)\rho g = 0.$$

Factor the left side.

$$-2\pi r\Delta r(\phi_{zz}|_{z=L} - \phi_{zz}|_{z=0}) - 2\pi L[(r+\Delta r)\phi_{rz}|_{r+\Delta r} - r\phi_{rz}|_{r}] + 2\pi r\Delta rL\rho g = 0$$

Divide both sides by $2\pi\Delta rL$.

$$-r\frac{\phi_{zz}|_{z=L} - \phi_{zz}|_{z=0}}{L} - \frac{(r+\Delta r)\phi_{rz}|_{r+\Delta r} - r\phi_{rz}|_{r}}{\Delta r} + \rho gr = 0$$

Take the limit as $\Delta r \to 0$.

$$-r\frac{\phi_{zz}|_{z=L} - \phi_{zz}|_{z=0}}{L} - \lim_{\Delta r\to 0}\frac{(r+\Delta r)\phi_{rz}|_{r+\Delta r} - r\phi_{rz}|_{r}}{\Delta r} + \rho gr = 0$$

The second term is the definition of the first derivative of $r\phi_{rz}$.

$$-r\frac{\phi_{zz}|_{z=L} - \phi_{zz}|_{z=0}}{L} - \frac{d}{dr}(r\phi_{rz}) + \rho gr = 0$$

Now substitute the expressions for ϕ_{rz} and ϕ_{zz}.

$$\begin{aligned}\phi_{rz} &= \tau_{rz} + \cancel{\rho v_r v_z} = \tau_{rz}\\ \phi_{zz} &= p\delta_{zz} + \cancel{\tau_{zz}} + \rho v_z v_z = p(z) + \rho v_z^2\end{aligned}$$

Since v_z does not depend on z, the ρv_z^2 terms cancel.

$$-r\frac{p|_{z=L} + \cancel{\rho v_z^2|_{z=L}} - p|_{z=0} - \cancel{\rho v_z^2|_{z=0}}}{L} - \frac{d}{dr}(r\tau_{rz}) + \rho gr = 0$$

Make it so that ρgr is in the fraction.

$$-r\frac{p|_{z=L} - p|_{z=0} - \rho gL}{L} - \frac{d}{dr}(r\tau_{rz}) = 0$$

Place $\rho g0$ in the numerator.

$$-r\frac{(p|_{z=L} - \rho gL) - (p|_{z=0} - \rho g0)}{L} - \frac{d}{dr}(r\tau_{rz}) = 0$$

The point of doing this is that now we can use the modified pressure $\mathscr{P}_z = p(z) - \rho g z$.

$$-r\frac{\mathscr{P}_L - \mathscr{P}_0}{L} - \frac{d}{dr}(r\tau_{rz}) = 0$$

So we have

$$\frac{d}{dr}(r\tau_{rz}) = \frac{\mathscr{P}_0 - \mathscr{P}_L}{L}r.$$

From Newton's law of viscosity we know that $\tau_{rz} = -\mu(dv_z/dr)$, so

$$\frac{d}{dr}\left(-\mu r\frac{dv_z}{dr}\right) = \frac{\mathscr{P}_0 - \mathscr{P}_L}{L}r.$$

Here we assume that the viscosity varies radially as a function of r, not axially as a function of z.

$$\mu = \mu(r)$$

One boundary condition is obtained from the assumption that the velocity is maximum furthest from the wall (at $r = 0$). The second boundary condition is obtained from the assumption that the fluid does not slip at the wall $r = R$.

$$\begin{aligned} \text{B.C. 1}: \quad & \frac{dv_z}{dr} = 0, \quad \text{at } r = 0 \\ \text{B.C. 2}: \quad & v_z = 0, \quad \text{at } r = R \end{aligned}$$

Integrate both sides of the differential equation with respect to r.

$$-\mu(r)r\frac{dv_z}{dr} = \frac{\mathscr{P}_0 - \mathscr{P}_L}{2L}r^2 + C_1$$

Apply the first boundary condition here to determine C_1.

$$-\mu(r)(0)\left.\frac{dv_z}{dr}\right|_{r=0} = \frac{\mathscr{P}_0 - \mathscr{P}_L}{2L}(0)^2 + C_1 \quad \rightarrow \quad 0 = C_1$$

Divide both sides by $-\mu(r)r$ to solve for dv_z/dr.

$$\frac{dv_z}{dr} = -\frac{\mathscr{P}_0 - \mathscr{P}_L}{2L}\frac{r}{\mu(r)}$$

Integrate both sides of the differential equation with respect to r once more.

$$v_z(r) = -\frac{\mathscr{P}_0 - \mathscr{P}_L}{2L}\int^r \frac{s}{\mu(s)}\,ds + C_2$$

The second boundary condition can be satisfied by choosing the lower limit of integration to be R and setting $C_2 = 0$. Use the minus sign to switch the limits of integration.

$$v_z(r) = \frac{\mathscr{P}_0 - \mathscr{P}_L}{2L}\int_r^R \frac{s}{\mu(s)}\,ds$$

Make the following substitution in the integral.

$$\begin{aligned} \bar{y} &= \frac{s}{R} \quad \rightarrow \quad s = R\bar{y} \\ d\bar{y} &= \frac{ds}{R} \quad \rightarrow \quad ds = R\,d\bar{y} \end{aligned}$$

We have

$$v_z(r) = \frac{\mathscr{P}_0 - \mathscr{P}_L}{2L} \int_{r/R}^{1} \frac{R\bar{y}}{\mu(R\bar{y})} \, (R \, d\bar{y}).$$

Let $y = r/R$.

$$v_z(y) = \frac{(\mathscr{P}_0 - \mathscr{P}_L)R^2}{2L} \int_y^1 \frac{\bar{y}}{\mu} \, d\bar{y}$$

Our aim now is to determine the average velocity. We do this by integrating the velocity over the cross-section that the fluid is flowing through and then dividing by that area.

$$\begin{aligned}
\langle v_z \rangle &= \frac{1}{A} \int v_z \, dA \\
&= \frac{1}{\pi R^2} \int_0^R v_z (2\pi r \, dr) \\
&= \frac{2}{R^2} \int_0^R r v_z \, dr
\end{aligned}$$

Let $y = r/R$.

$$\begin{aligned}
&= \frac{2}{R^2} \int_0^1 (Ry) v_z \left(\frac{dr}{dy} \, dy \right) \\
&= \frac{2}{R^2} \int_0^1 (Ry) v_z \, (R \, dy) \\
&= 2 \int_0^1 y v_z \, dy \\
&= 2 \int_0^1 y \left[\frac{(\mathscr{P}_0 - \mathscr{P}_L)R^2}{2L} \int_y^1 \frac{\bar{y}}{\mu(\bar{y})} \, d\bar{y} \right] dy \\
&= \frac{(\mathscr{P}_0 - \mathscr{P}_L)R^2}{L} \int_0^1 \int_y^1 y \frac{\bar{y}}{\mu(\bar{y})} \, d\bar{y} \, dy \\
&= \frac{(\mathscr{P}_0 - \mathscr{P}_L)R^2}{L} \int_0^1 \int_0^{\bar{y}} y \frac{\bar{y}}{\mu(\bar{y})} \, dy \, d\bar{y} \\
&= \frac{(\mathscr{P}_0 - \mathscr{P}_L)R^2}{L} \left(\int_0^1 \frac{\bar{y}}{\mu(\bar{y})} \, d\bar{y} \right) \left(\int_0^{\bar{y}} y \, dy \right) \\
&= \frac{(\mathscr{P}_0 - \mathscr{P}_L)R^2}{L} \left(\int_0^1 \frac{\bar{y}}{\mu(\bar{y})} \, d\bar{y} \right) \left(\frac{\bar{y}^2}{2} \right)
\end{aligned}$$

The average velocity is thus

$$\langle v_z \rangle = \frac{(\mathscr{P}_0 - \mathscr{P}_L)R^2}{2L} \int_0^1 \frac{\bar{y}^3}{\mu} \, d\bar{y}.$$

Therefore,

$$\frac{v_z}{\langle v_z \rangle} = \frac{\cancel{\frac{(\mathscr{P}_0 - \mathscr{P}_L)R^2}{2L}} \int_y^1 \frac{\bar{y}}{\mu} \, d\bar{y}}{\cancel{\frac{(\mathscr{P}_0 - \mathscr{P}_L)R^2}{2L}} \int_0^1 \frac{\bar{y}^3}{\mu} \, d\bar{y}} = \frac{\int_y^1 (\bar{y}/\mu) d\bar{y}}{\int_0^1 (\bar{y}^3/\mu) d\bar{y}}.$$