Problem 3A.1

Torque required to turn a friction bearing (Fig. 3A.1). Calculate the required torque in $\text{lb}_f \cdot \text{ft}$ and power consumption in horsepower to turn the shaft in the friction bearing shown in the figure. The length of the bearing surface on the shaft is 2 in, and the shaft is rotating at 200 rpm. The viscosity of the lubricant is 200 cp, and its density is 50 lb_m/ft^3. Neglect the effect of eccentricity.
Answers: 0.32 $\text{lb}_f \cdot \text{ft}$; 0.012 hp = 0.009 kW

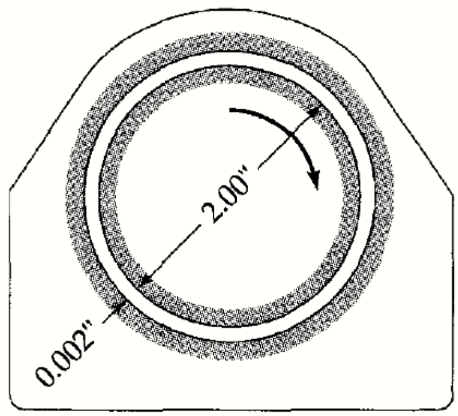

Fig. 3A.1. Friction bearing.

Solution

The first step is to determine the velocity profile v_θ in the annular space of the bearing. This allows us to calculate the viscous stress $\tau_{r\theta}$, which then allows us to find the torque acting on the shaft. We assume that the fluid flows only in the θ-direction and that the velocity varies as a function of radius only.

$$\mathbf{v} = v_\theta(r)\hat{\boldsymbol{\theta}}$$

If we assume the fluid does not slip on the walls, then it has the wall's velocity at $r = \kappa R$ and $r = R$. The tangential velocity is obtained by multiplying the angular velocity by the moment arm.

Boundary Condition 1: $v_\theta(\kappa R) = \Omega_i \kappa R$
Boundary Condition 2: $v_\theta(R) = 0$

The equation of continuity results by considering a mass balance over a volume element that the fluid is flowing through. Assuming the fluid density ρ is constant, the equation simplifies to

$$\nabla \cdot \mathbf{v} = 0. \tag{1}$$

The equation of motion results by considering a momentum balance over a volume element that the fluid is flowing through. Assuming the fluid viscosity μ is also constant, the equation simplifies to the Navier-Stokes equation.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla \cdot \rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g} \tag{2}$$

As this is a vector equation, it actually represents three scalar equations—one for each variable in the chosen coordinate system. Using cylindrical coordinates is the appropriate choice for this problem, so equations (1) and (2) will be used in (r, θ, z). From Appendix B.4 on page 846, the continuity equation becomes

$$\underbrace{\frac{1}{r}\frac{\partial}{\partial r}(rv_r)}_{=0} + \underbrace{\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}}_{=0} + \underbrace{\frac{\partial v_z}{\partial z}}_{=0} = 0,$$

which doesn't tell us anything. From Appendix B.6 on page 848, the Navier-Stokes equation yields the following three scalar equations in cylindrical coordinates.

$$\rho\left(\underbrace{\frac{\partial v_r}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_r}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_r}{\partial z}}_{=0} - \frac{v_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left[\underbrace{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)}_{=0} + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_r}{\partial z^2}}_{=0} - \underbrace{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}}_{=0}\right] + \underbrace{\rho g_r}_{=0}$$

$$\rho\left(\underbrace{\frac{\partial v_\theta}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_\theta}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_\theta}{\partial z}}_{=0} + \underbrace{\frac{v_r v_\theta}{r}}_{=0}\right) = \underbrace{-\frac{1}{r}\frac{\partial p}{\partial \theta}}_{=0} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_\theta}{\partial z^2}}_{=0} + \underbrace{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}}_{=0}\right] + \underbrace{\rho g_\theta}_{=0}$$

$$\rho\left(\underbrace{\frac{\partial v_z}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_z}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_z}{\partial z}}_{=0}\right) = -\frac{\partial p}{\partial z} + \mu\left[\underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)}_{=0} + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_z}{\partial z^2}}_{=0}\right] + \rho g_z$$

The relevant equation for the velocity is the θ-equation, which has simplified considerably from the assumption that $\mathbf{v} = v_\theta(r)\hat{\boldsymbol{\theta}}$.

$$0 = \mu\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rv_\theta)\right)$$

Divide both sides by μ.

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rv_\theta)\right) = 0$$

Integrate both sides with respect to r.

$$\frac{1}{r}\frac{d}{dr}(rv_\theta) = C_1$$

Multiply both sides by r.

$$\frac{d}{dr}(rv_\theta) = C_1 r$$

Integrate both sides with respect to r once more.

$$rv_\theta = C_1\frac{r^2}{2} + C_2$$

Divide both sides by r.

$$v_\theta(r) = C_1\frac{r}{2} + \frac{C_2}{r}$$

Apply the two boundary conditions here to determine C_1 and C_2.

$$v_\theta(\kappa R) = C_1\frac{\kappa R}{2} + \frac{C_2}{\kappa R} = \Omega_i \kappa R$$

$$v_\theta(R) = C_1\frac{R}{2} + \frac{C_2}{R} = 0$$

Solving the system of equations yields

$$C_1 = -2\frac{\kappa^2}{1-\kappa^2}\Omega_i \quad \text{and} \quad C_2 = \frac{\kappa^2 R^2}{1-\kappa^2}\Omega_i.$$

We then have

$$\begin{aligned} v_\theta(r) &= -2\frac{\kappa^2}{1-\kappa^2}\Omega_i \frac{r}{2} + \frac{\kappa^2 R^2}{1-\kappa^2}\frac{\Omega_i}{r} \\ &= \frac{\kappa^2}{1-\kappa^2}\left(-r + \frac{R^2}{r}\right)\Omega_i \\ &= \frac{\kappa^2}{1-\kappa^2}\left(\frac{R^2 - r^2}{r}\right)\Omega_i. \end{aligned}$$

The torque on the inner cylinder is obtained by multiplying $(-\tau_{r\theta})|_{r=\kappa R}$, the viscous force per unit area in the θ-direction on a plane perpendicular to the r-direction, by the lateral surface area of the cylinder by the moment arm. The minus sign in front of $\tau_{r\theta}$ indicates that the fluid is at a higher radius than the cylinder it is acting upon. The expression for $\tau_{r\theta}$ is given in Table B.1 on page 844.

$$\begin{aligned} T_i &= (-\tau_{r\theta})|_{r=\kappa R} \cdot 2\pi(\kappa R)L \cdot \kappa R \\ &= \mu r \frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\bigg|_{r=\kappa R} \cdot 2\pi\kappa^2 R^2 L \\ &= -\frac{2\mu\kappa^2 R^2}{r^2(1-\kappa^2)}\Omega_i\bigg|_{r=\kappa R} \cdot 2\pi\kappa^2 R^2 L \end{aligned}$$

Thus, the torque exerted by the fluid on the inner cylinder is

$$T_i = -\frac{4\pi\mu L\kappa^2 R^2}{1-\kappa^2}\Omega_i.$$

The torque that needs to be applied to the inner cylinder to maintain the motion is therefore

$$\boxed{T_{ai} = \frac{4\pi\mu L\kappa^2 R^2}{1-\kappa^2}\Omega_i.}$$

The relevant numbers in SI units are

$$\begin{aligned} \Omega_i &= 200\,\frac{\cancel{\text{revolutions}}}{\cancel{\text{minute}}} \times \frac{2\pi \text{ radians}}{1\,\cancel{\text{revolution}}} \times \frac{1\,\cancel{\text{minute}}}{60 \text{ seconds}} = \frac{20\pi}{3}\frac{\text{radians}}{\text{second}} \\ L &= 2\,\cancel{\text{in}} \times \frac{2.54\,\cancel{\text{cm}}}{1\,\cancel{\text{in}}} \times \frac{1 \text{ m}}{100\,\cancel{\text{cm}}} = 0.0508 \text{ m} \\ R &= 1.002\,\cancel{\text{in}} \times \frac{2.54\,\cancel{\text{cm}}}{1\,\cancel{\text{in}}} \times \frac{1 \text{ m}}{100\,\cancel{\text{cm}}} = 0.0254508 \text{ m} \\ \mu &= 200\,\cancel{\text{centipoise}} \times \frac{10^{-3} \text{ Pa}\cdot\text{s}}{1\,\cancel{\text{centipoise}}} = 0.2 \text{ Pa}\cdot\text{s} \\ \kappa &= \frac{\kappa R}{R} = \frac{1\,\cancel{\text{in}}}{1.002\,\cancel{\text{in}}} \approx 0.998004. \end{aligned}$$

Plugging in the numbers, we get (the following conversion factors are found on pages 869 and 870)

$$T_{ai} \approx 0.432585\,\cancel{\text{J}} \times \frac{7.3756 \times 10^{-1} \text{ lb}_f \cdot \text{ft}}{1\,\cancel{\text{J}}} \approx 0.32 \text{ lb}_f \cdot \text{ft}.$$

The power is obtained by multiplying the torque by the angular velocity.

$$P = (0.432585 \text{ J})\left(\frac{20\pi}{3}\frac{\text{radians}}{\text{second}}\right) \approx 9\,\cancel{\text{Watts}} \times \frac{1\,\cancel{\text{kW}}}{1000\,\cancel{\text{Watts}}} \times \frac{1.3410 \text{ hp}}{1\,\cancel{\text{kW}}} \approx 0.012 \text{ hp}$$