Problem 3B.1

Flow between coaxial cylinders and concentric spheres.

(a) The space between two coaxial cylinders is filled with an incompressible fluid at constant temperature. The radii of the inner and outer wetted surfaces are κR and R, respectively. The angular velocities of rotation of the inner and outer cylinders are Ω_i and Ω_o. Determine the velocity distribution in the fluid and the torques exerted by the fluid on the two cylinders needed to maintain the motion.

(b) Repeat part (a) for two concentric spheres.

Answers:

$$\textbf{(a) } v_\theta = \frac{\kappa R}{1-\kappa^2}\left[(\Omega_o - \Omega_i\kappa^2)\left(\frac{r}{\kappa R}\right) + (\Omega_i - \Omega_o)\left(\frac{\kappa R}{r}\right)\right]$$

$$\textbf{(b) } v_\phi = \frac{\kappa R}{1-\kappa^3}\left[(\Omega_o - \Omega_i\kappa^3)\left(\frac{r}{\kappa R}\right) + (\Omega_i - \Omega_o)\left(\frac{\kappa R}{r}\right)^2\right]\sin\theta$$

Solution

Part (a)

We assume that the fluid flows only in the θ-direction and that the velocity varies as a function of radius only.

$$\mathbf{v} = v_\theta(r)\hat{\boldsymbol{\theta}}$$

If we assume the fluid does not slip on the walls, then it has the wall's velocity at $r = \kappa R$ and $r = R$. The tangential velocity is obtained by multiplying the angular velocity by the distance from the axis of rotation (the moment arm).

Boundary Condition 1: $v_\theta(\kappa R) = \Omega_i \kappa R$

Boundary Condition 2: $v_\theta(R) = \Omega_o R$

The equation of continuity results by considering a mass balance over a volume element that the fluid is flowing through. Assuming the fluid density ρ is constant, the equation simplifies to

$$\nabla \cdot \mathbf{v} = 0. \tag{1}$$

The equation of motion results by considering a momentum balance over a volume element that the fluid is flowing through. Assuming the fluid viscosity μ is also constant, the equation simplifies to the Navier-Stokes equation.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla \cdot \rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g} \tag{2}$$

As this is a vector equation, it actually represents three scalar equations—one for each variable in the chosen coordinate system. Using cylindrical coordinates is the appropriate choice for this problem, so equations (1) and (2) will be used in (r, θ, z). From Appendix B.4 on page 846, the continuity equation becomes

$$\underbrace{\frac{1}{r}\frac{\partial}{\partial r}(rv_r)}_{=0} + \underbrace{\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}}_{=0} + \underbrace{\frac{\partial v_z}{\partial z}}_{=0} = 0,$$

which doesn't tell us anything. From Appendix B.6 on page 848, the Navier-Stokes equation yields the following three scalar equations in cylindrical coordinates.

$$\rho\bigg(\underbrace{\frac{\partial v_r}{\partial t}}_{=0}+\underbrace{v_r\frac{\partial v_r}{\partial r}}_{=0}+\underbrace{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}}_{=0}+\underbrace{v_z\frac{\partial v_r}{\partial z}}_{=0}-\frac{v_\theta^2}{r}\bigg)=-\frac{\partial p}{\partial r}+\mu\bigg[\underbrace{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)}_{=0}+\underbrace{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}}_{=0}+\underbrace{\frac{\partial^2 v_r}{\partial z^2}}_{=0}-\underbrace{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}}_{=0}\bigg]+\underbrace{\rho g_r}_{=0}$$

$$\rho\bigg(\underbrace{\frac{\partial v_\theta}{\partial t}}_{=0}+\underbrace{v_r\frac{\partial v_\theta}{\partial r}}_{=0}+\underbrace{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}}_{=0}+\underbrace{v_z\frac{\partial v_\theta}{\partial z}}_{=0}+\underbrace{\frac{v_r v_\theta}{r}}_{=0}\bigg)=\underbrace{-\frac{1}{r}\frac{\partial p}{\partial \theta}}_{=0}+\mu\bigg[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right)+\underbrace{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}}_{=0}+\underbrace{\frac{\partial^2 v_\theta}{\partial z^2}}_{=0}+\underbrace{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}}_{=0}\bigg]+\underbrace{\rho g_\theta}_{=0}$$

$$\rho\bigg(\underbrace{\frac{\partial v_z}{\partial t}}_{=0}+\underbrace{v_r\frac{\partial v_z}{\partial r}}_{=0}+\underbrace{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}}_{=0}+\underbrace{v_z\frac{\partial v_z}{\partial z}}_{=0}\bigg)=-\frac{\partial p}{\partial z}+\mu\bigg[\underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)}_{=0}+\underbrace{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}}_{=0}+\underbrace{\frac{\partial^2 v_z}{\partial z^2}}_{=0}\bigg]+\rho g_z$$

The relevant equation for the velocity is the θ-equation, which has simplified considerably from the assumption that $\mathbf{v}=v_\theta(r)\hat{\boldsymbol{\theta}}$.

$$0=\mu\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rv_\theta)\right)$$

Divide both sides by μ.

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rv_\theta)\right)=0$$

Integrate both sides with respect to r.

$$\frac{1}{r}\frac{d}{dr}(rv_\theta)=C_1$$

Multiply both sides by r.

$$\frac{d}{dr}(rv_\theta)=C_1 r$$

Integrate both sides with respect to r once more.

$$rv_\theta=C_1\frac{r^2}{2}+C_2$$

Divide both sides by r.

$$v_\theta(r)=C_1\frac{r}{2}+\frac{C_2}{r}$$

Apply the two boundary conditions here to determine C_1 and C_2.

$$v_\theta(\kappa R)=C_1\frac{\kappa R}{2}+\frac{C_2}{\kappa R}=\Omega_i\kappa R$$

$$v_\theta(R)=C_1\frac{R}{2}+\frac{C_2}{R}=\Omega_o R$$

Solving the system of equations yields

$$C_1=\frac{2(\Omega_o-\kappa^2\Omega_i)}{1-\kappa^2}\quad\text{and}\quad C_2=\frac{\kappa^2R^2(\Omega_i-\Omega_o)}{1-\kappa^2}.$$

We then have

$$v_\theta(r)=\frac{2(\Omega_o-\kappa^2\Omega_i)}{1-\kappa^2}\frac{r}{2}+\frac{\kappa^2R^2(\Omega_i-\Omega_o)}{1-\kappa^2}\frac{1}{r}.$$

Therefore, upon factoring $\kappa R/(1-\kappa^2)$,

$$\boxed{v_\theta(r) = \frac{\kappa R}{1-\kappa^2}\left[(\Omega_o - \Omega_i\kappa^2)\left(\frac{r}{\kappa R}\right) + (\Omega_i - \Omega_o)\left(\frac{\kappa R}{r}\right)\right].}$$

The torque on the inner cylinder is obtained by multiplying $(-\tau_{r\theta})|_{r=\kappa R}$, the viscous force per unit area in the θ-direction on a plane perpendicular to the r-direction, by the lateral surface area of the cylinder by the moment arm. The minus sign in front of $\tau_{r\theta}$ indicates that the fluid is at a higher radius than the cylinder it is acting upon. The expression for $\tau_{r\theta}$ is given in Table B.1 on page 844.

$$\begin{aligned} T_i &= (-\tau_{r\theta})|_{r=\kappa R} \cdot 2\pi(\kappa R)L \cdot \kappa R \\ &= \mu r \frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\bigg|_{r=\kappa R} \cdot 2\pi\kappa^2 R^2 L \\ &= \frac{2\mu\kappa^2 R^2(\Omega_o - \Omega_i)}{r^2(1-\kappa^2)}\bigg|_{r=\kappa R} \cdot 2\pi\kappa^2 R^2 L \end{aligned}$$

Thus, the torque exerted by the fluid on the inner cylinder is

$$T_i = \frac{4\pi\mu L\kappa^2 R^2(\Omega_o - \Omega_i)}{1-\kappa^2}.$$

The torque that needs to be applied to the inner cylinder to maintain the motion is therefore

$$\boxed{T_{ai} = -\frac{4\pi\mu L\kappa^2 R^2(\Omega_o - \Omega_i)}{1-\kappa^2}.}$$

Similarly, the torque on the outer cylinder is

$$T_o = (\tau_{r\theta})|_{r=R} \cdot 2\pi RL \cdot R.$$

The sign in front of $\tau_{r\theta}$ is positive here because the fluid is acting on the outer cylinder, which is at a higher radius than the fluid.

$$\begin{aligned} &= -\mu r \frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\bigg|_{r=R} \cdot 2\pi R^2 L \\ &= -\frac{2\mu\kappa^2 R^2(\Omega_o - \Omega_i)}{r^2(1-\kappa^2)}\bigg|_{r=R} \cdot 2\pi R^2 L \end{aligned}$$

Thus, the torque exerted by the fluid on the outer cylinder is

$$T_o = -\frac{4\pi\mu L\kappa^2 R^2(\Omega_o - \Omega_i)}{1-\kappa^2}.$$

The torque that needs to be applied to the outer cylinder to maintain the motion is therefore

$$\boxed{T_{ao} = \frac{4\pi\mu L\kappa^2 R^2(\Omega_o - \Omega_i)}{1-\kappa^2}.}$$

Part (b)

For two concentric spheres a spherical coordinate system (r, θ, ϕ) is used, where θ represents the angle from the polar axis. We assume that the fluid flows only in the ϕ-direction and that the velocity varies with the distance from the axis of rotation (that is, as a function of r and θ).

$$\mathbf{v} = v_\phi(r, \theta)\hat{\boldsymbol{\phi}}$$

If we assume the fluid does not slip on the walls, then it has the wall's velocity at $r = \kappa R$ and $r = R$. The tangential velocity is obtained by multiplying the angular velocity by the distance from the axis of rotation (the moment arm).

$$\begin{aligned} &\text{Boundary Condition 1:} \qquad v_\phi(\kappa R, \theta) = \Omega_i \kappa R \sin\theta \\ &\text{Boundary Condition 2:} \qquad v_\phi(R, \theta) = \Omega_o R \sin\theta \end{aligned}$$

The equation of continuity results by considering a mass balance over a volume element that the fluid is flowing through. Assuming the fluid density ρ is constant, the equation simplifies to

$$\nabla \cdot \mathbf{v} = 0. \tag{3}$$

The equation of motion results by considering a momentum balance over a volume element that the fluid is flowing through. Assuming the fluid viscosity μ is also constant, the equation simplifies to the Navier-Stokes equation.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla \cdot \rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g} \tag{4}$$

As this is a vector equation, it actually represents three scalar equations—one for each variable in the chosen coordinate system. From Appendix B.4 on page 846, the continuity equation in spherical coordinates is

$$\underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r)}_{=0} + \underbrace{\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta)}_{=0} + \underbrace{\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}}_{=0} = 0,$$

which doesn't tell us anything. From Appendix B.6 on page 848, the Navier-Stokes equation yields the following three scalar equations in spherical coordinates.

$$\begin{aligned} \rho\left(\underbrace{\frac{\partial v_r}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_r}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta}}_{=0} + \underbrace{\frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial\phi}}_{=0} - \frac{\overbrace{v_\theta^2}^{=0} + v_\phi^2}{r}\right) &= -\frac{\partial p}{\partial r} \\ &+ \mu\left[\underbrace{\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 v_r)}_{=0} + \underbrace{\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right)}_{=0} + \underbrace{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2}}_{=0}\right] + \rho g_r \end{aligned}$$

$$\rho\bigg(\underbrace{\frac{\partial v_\theta}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_\theta}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}}_{=0} + \underbrace{\frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}}_{=0} + \frac{\overbrace{v_r v_\theta}^{=0} - v_\phi^2\cot\theta}{r}\bigg) = -\frac{1}{r}\frac{\partial p}{\partial \theta}$$
$$+ \mu\bigg[\underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right)}_{=0} + \underbrace{\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(v_\theta\sin\theta)\right)}_{=0}$$
$$+ \underbrace{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial \phi^2}}_{=0} + \underbrace{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}}_{=0} - \underbrace{\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}}_{=0}\bigg] + \rho g_\theta$$

$$\rho\bigg(\underbrace{\frac{\partial v_\phi}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_\phi}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta}}_{=0} + \underbrace{\frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}}_{=0} + \underbrace{\frac{v_\phi v_r + v_\theta v_\phi\cot\theta}{r}}_{=0}\bigg) = \underbrace{-\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}}_{=0}$$
$$+ \mu\bigg[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(v_\phi\sin\theta)\right)$$
$$+ \underbrace{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial \phi^2}}_{=0} + \underbrace{\frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial \phi}}_{=0} + \underbrace{\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\theta}{\partial \phi}}_{=0}\bigg] + \underbrace{\rho g_\phi}_{=0}$$

The relevant equation for the velocity is the ϕ-equation, which has simplified considerably from the assumption that $\mathbf{v} = v_\phi(r,\theta)\hat{\boldsymbol{\phi}}$.

$$0 = \mu\bigg[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(v_\phi\sin\theta)\right)\bigg]$$

Multiply both sides by r^2/μ.

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(v_\phi\sin\theta)\right) = 0$$

From the boundary conditions, we hypothesize that the solution is of the form $v_\phi(r,\theta) = f(r)\sin\theta$. Substitute this into the PDE

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}[f(r)\sin\theta]\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}[f(r)\sin^2\theta]\right) = 0$$

and the boundary conditions.

$$v_\phi(\kappa R,\theta) = f(\kappa R)\sin\theta = \Omega_i\kappa R\sin\theta \quad \rightarrow \quad f(\kappa R) = \Omega_i\kappa R$$
$$v_\phi(R,\theta) = f(R)\sin\theta = \Omega_o R\sin\theta \quad \rightarrow \quad f(R) = \Omega_o R$$

Expand the left side of the PDE.

$$(2rf' + r^2f'')\sin\theta + (-2\sin\theta)f = 0$$

Divide both sides by $\sin\theta$.

$$r^2f'' + 2rf' - 2f = 0$$

Because θ no longer appears in the equation, the hypothesis made about the form of the solution is legitimate. What we have now is an equidimensional ODE for f, so it has solutions of the form $f(r) = r^n$. Find the derivatives of it

$$f(r) = r^n \quad \rightarrow \quad f'(r) = nr^{n-1} \quad \rightarrow \quad f''(r) = n(n-1)r^{n-2}$$

and substitute these expressions into the ODE to determine n.

$$n(n-1)r^n + 2nr^n - 2r^n = 0$$

$$n(n-1) + 2n - 2 = 0$$
$$n^2 + n - 2 = 0$$
$$(n+2)(n-1) = 0$$
$$n = \{-2, 1\}$$

Consequently,

$$f(r) = C_3 r + \frac{C_4}{r^2}.$$

Use the boundary conditions for f to determine C_3 and C_4.

$$f(\kappa R) = C_3 \kappa R + \frac{C_4}{\kappa^2 R^2} = \Omega_i \kappa R$$
$$f(R) = C_3 R + \frac{C_4}{R^2} = \Omega_o R$$

Solving this system of equations gives

$$C_3 = \frac{\Omega_o - \kappa^3 \Omega_i}{1 - \kappa^3} \quad \text{and} \quad C_4 = \frac{\kappa^3 R^3 (\Omega_i - \Omega_o)}{1 - \kappa^3}.$$

We then have

$$\begin{aligned} f(r) &= \frac{\Omega_o - \kappa^3 \Omega_i}{1 - \kappa^3} r + \frac{\kappa^3 R^3 (\Omega_i - \Omega_o)}{1 - \kappa^3} \frac{1}{r^2} \\ &= \frac{\kappa R}{1 - \kappa^3} \left[(\Omega_o - \kappa^3 \Omega_i) \left(\frac{r}{\kappa R} \right) + (\Omega_i - \Omega_o) \left(\frac{\kappa R}{r} \right)^2 \right]. \end{aligned}$$

Therefore,

$$\boxed{v_\phi(r, \theta) = \frac{\kappa R}{1 - \kappa^3} \left[(\Omega_o - \Omega_i \kappa^3) \left(\frac{r}{\kappa R} \right) + (\Omega_i - \Omega_o) \left(\frac{\kappa R}{r} \right)^2 \right] \sin\theta.}$$

Since v_ϕ depends on θ, the torque on the inner sphere is obtained by integrating $(-\tau_{r\phi})|_{r=\kappa R} \cdot \kappa R \sin\theta$, the product of the moment arm and the viscous force per unit area in the ϕ-direction on a plane perpendicular to the r-direction, over the sphere's surface area. The minus sign in front of $\tau_{r\phi}$ indicates that the fluid is at a higher radius than the sphere it is acting upon.

The expression for $\tau_{r\phi}$ is given in Table B.1 on page 844.

$$\begin{aligned}
T_i &= \iint_{S_i} (-\tau_{r\phi})|_{r=\kappa R} \cdot \kappa R \sin\theta \, dS \\
&= \int_0^{2\pi} \int_0^{\pi} (-\tau_{r\phi})|_{r=\kappa R} \cdot \kappa R \sin\theta [(\kappa R)^2 \sin\theta \, d\theta \, d\phi] \\
&= \int_0^{2\pi} \int_0^{\pi} \mu r \frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\bigg|_{r=\kappa R} \cdot \kappa^3 R^3 \sin^2\theta \, d\theta \, d\phi \\
&= \int_0^{2\pi} \int_0^{\pi} \frac{3\mu\kappa^3 R^3(\Omega_o - \Omega_i)}{r^3(1-\kappa^3)} \sin\theta\bigg|_{r=\kappa R} \cdot \kappa^3 R^3 \sin^2\theta \, d\theta \, d\phi \\
&= \int_0^{2\pi} \int_0^{\pi} \frac{3\mu(\Omega_o - \Omega_i)}{1-\kappa^3} \sin\theta \cdot \kappa^3 R^3 \sin^2\theta \, d\theta \, d\phi \\
&= \frac{3\mu\kappa^3 R^3(\Omega_o - \Omega_i)}{1-\kappa^3} \left(\int_0^{2\pi} d\phi\right)\left(\int_0^{\pi} \sin^3\theta \, d\theta\right) \\
&= \frac{3\mu\kappa^3 R^3(\Omega_o - \Omega_i)}{1-\kappa^3} (2\pi)\left(\frac{4}{3}\right)
\end{aligned}$$

Thus, the torque exerted by the fluid on the inner sphere is

$$T_i = \frac{8\pi\mu\kappa^3 R^3(\Omega_o - \Omega_i)}{1-\kappa^3}.$$

The torque that needs to be applied to the inner sphere to maintain the motion is therefore

$$\boxed{T_{ai} = -\frac{8\pi\mu\kappa^3 R^3(\Omega_o - \Omega_i)}{1-\kappa^3}.}$$

Similarly, the torque on the outer sphere is

$$T_o = \iint_{S_o} (\tau_{r\phi})|_{r=R} \cdot R \sin\theta \, dS$$

The sign in front of $\tau_{r\phi}$ is positive here because the fluid is acting on the outer sphere, which is at a higher radius than the fluid.

$$\begin{aligned}
&= \int_0^{2\pi} \int_0^{\pi} (\tau_{r\phi})|_{r=R} \cdot R \sin\theta (R^2 \sin\theta \, d\theta \, d\phi) \\
&= \int_0^{2\pi} \int_0^{\pi} -\mu r \frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\bigg|_{r=R} \cdot R^3 \sin^2\theta \, d\theta \, d\phi \\
&= \int_0^{2\pi} \int_0^{\pi} -\frac{3\mu\kappa^3 R^3(\Omega_o - \Omega_i)}{r^3(1-\kappa^3)} \sin\theta\bigg|_{r=R} \cdot R^3 \sin^2\theta \, d\theta \, d\phi \\
&= \int_0^{2\pi} \int_0^{\pi} -\frac{3\mu\kappa^3(\Omega_o - \Omega_i)}{1-\kappa^3} \sin\theta \cdot R^3 \sin^2\theta \, d\theta \, d\phi \\
&= -\frac{3\mu\kappa^3 R^3(\Omega_o - \Omega_i)}{1-\kappa^3} \left(\int_0^{2\pi} d\phi\right)\left(\int_0^{\pi} \sin^3\theta \, d\theta\right) \\
&= -\frac{3\mu\kappa^3 R^3(\Omega_o - \Omega_i)}{1-\kappa^3} (2\pi)\left(\frac{4}{3}\right)
\end{aligned}$$

Thus, the torque exerted by the fluid on the outer sphere is

$$T_o = -\frac{8\pi\mu\kappa^3 R^3(\Omega_o - \Omega_i)}{1-\kappa^3}.$$

The torque that needs to be applied to the outer sphere to maintain the motion is therefore

$$\boxed{T_{ao} = \frac{8\pi\mu\kappa^3 R^3(\Omega_o - \Omega_i)}{1-\kappa^3}.}$$