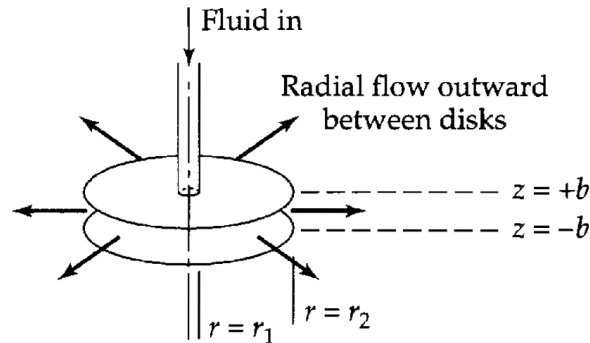


### Problem 3B.10

**Radial flow between parallel disks** (Fig. 3B.10). A part of a lubrication system consists of two circular disks between which a lubricant flows radially. The flow takes place because of a modified pressure difference  $\mathcal{P}_1 - \mathcal{P}_2$  between the inner and outer radii  $r_1$  and  $r_2$ , respectively.



**Fig. 3B.10.** Outward radial flow in the space between two parallel, circular disks.

- (a) Write the equations of continuity and motion for this flow system, assuming steady-state, laminar, incompressible Newtonian flow. Consider only the region  $r_1 \leq r \leq r_2$  and a flow that is radially directed.

- (b) Show how the equation of continuity enables one to simplify the equation of motion to give

$$-\rho \frac{\phi^2}{r^3} = -\frac{d\mathcal{P}}{dr} + \mu \frac{1}{r} \frac{d^2\phi}{dz^2} \quad (3B.10-1)$$

in which  $\phi = rv_r$  is a function of  $z$  only. Why is  $\phi$  independent of  $r$ ?

- (c) It can be shown that no solution exists for Eq. 3B.10-1 unless the nonlinear term containing  $\phi$  is omitted. Omission of this term corresponds to the “creeping flow assumption.” Show that for creeping flow, Eq. 3B.10-1 can be integrated with respect to  $r$  to give

$$0 = (\mathcal{P}_1 - \mathcal{P}_2) + \left( \mu \ln \frac{r_2}{r_1} \right) \frac{d^2\phi}{dz^2} \quad (3B.10-2)$$

- (d) Show that further integration with respect to  $z$  gives

$$v_r(r, z) = \frac{(\mathcal{P}_1 - \mathcal{P}_2)b^2}{2\mu r \ln(r_2/r_1)} \left[ 1 - \left( \frac{z}{b} \right)^2 \right] \quad (3B.10-3)$$

- (e) Show that the mass flow rate is

$$w = \frac{4\pi(\mathcal{P}_1 - \mathcal{P}_2)b^3\rho}{3\mu \ln(r_2/r_1)} \quad (3B.10-4)$$

- (f) Sketch the curves  $\mathcal{P}(r)$  and  $v_r(r, z)$ .

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### Solution

**Part (a)**

Since the flow is radial and there is space between the disks, we assume that the velocity varies as a function of radius and height and that the fluid moves only in the  $r$ -direction.

$$\mathbf{v} = v_r(r, z)\hat{\mathbf{r}}$$

If we assume the fluid does not slip on the walls, then it has the wall's velocity at  $z = -b$  and  $z = b$ .

$$\text{Boundary Condition 1: } v_r(r, -b) = 0$$

$$\text{Boundary Condition 2: } v_r(r, b) = 0$$

The equation of continuity results by considering a mass balance over a volume element that the fluid is flowing through. Assuming the fluid density  $\rho$  is constant, the equation simplifies to

$$\nabla \cdot \mathbf{v} = 0. \quad (1)$$

The equation of motion results by considering a momentum balance over a volume element that the fluid is flowing through. Assuming the fluid viscosity  $\mu$  is also constant, the equation simplifies to the Navier-Stokes equation.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla \cdot \rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g} \quad (2)$$

As this is a vector equation, it actually represents three scalar equations—one for each variable in the chosen coordinate system. Using cylindrical coordinates is the appropriate choice for this problem, so equations (1) and (2) will be used in  $(r, \theta, z)$ . From Appendix B.4 on page 846, the continuity equation becomes

$$\frac{1}{r} \frac{\partial}{\partial r}(rv_r) + \underbrace{\frac{1}{r} \frac{\partial v_\theta}{\partial \theta}}_{=0} + \underbrace{\frac{\partial v_z}{\partial z}}_{=0} = 0.$$

Multiply both sides by  $r$ .

$$\frac{\partial}{\partial r}(rv_r) = 0$$

It is here we see that  $rv_r$  is independent of  $r$ . Integrate both sides partially with respect to  $r$ .

$$rv_r = \phi(z),$$

where  $\phi$  is an arbitrary function of  $z$ . From Appendix B.6 on page 848, the Navier-Stokes equation yields the following three scalar equations in cylindrical coordinates.

$$\begin{aligned} \rho \left( \underbrace{\frac{\partial v_r}{\partial t}}_{=0} + v_r \frac{\partial v_r}{\partial r} + \underbrace{\frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta}}_{=0} + \underbrace{v_z \frac{\partial v_r}{\partial z}}_{=0} - \underbrace{\frac{v_\theta^2}{r}}_{=0} \right) &= -\frac{\partial p}{\partial r} + \mu \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r}(rv_r) \right) + \underbrace{\frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2}}_{=0} + \frac{\partial^2 v_r}{\partial z^2} - \underbrace{\frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta}}_{=0} \right] + \rho g_r \\ \rho \left( \underbrace{\frac{\partial v_\theta}{\partial t}}_{=0} + v_r \frac{\partial v_\theta}{\partial r} + \underbrace{\frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta}}_{=0} + \underbrace{v_z \frac{\partial v_\theta}{\partial z}}_{=0} + \underbrace{\frac{v_r v_\theta}{r}}_{=0} \right) &= -\underbrace{\frac{1}{r} \frac{\partial p}{\partial \theta}}_{=0} + \mu \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r}(rv_\theta) \right) + \underbrace{\frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2}}_{=0} + \frac{\partial^2 v_\theta}{\partial z^2} + \underbrace{\frac{2}{r^2} \frac{\partial v_r}{\partial \theta}}_{=0} \right] + \underbrace{\rho g_\theta}_{=0} \\ \rho \left( \underbrace{\frac{\partial v_z}{\partial t}}_{=0} + v_r \frac{\partial v_z}{\partial r} + \underbrace{\frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta}}_{=0} + \underbrace{v_z \frac{\partial v_z}{\partial z}}_{=0} \right) &= -\underbrace{\frac{\partial p}{\partial z}}_{=0} + \mu \left[ \underbrace{\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right)}_{=0} + \underbrace{\frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_z}{\partial z^2}}_{=0} \right] + \underbrace{\rho g_z}_{=0} \end{aligned}$$

The relevant equation for the velocity is the  $r$ -equation, which has simplified considerably from the assumption that  $\mathbf{v} = v_r(r, z)\hat{\mathbf{r}}$ .

$$\rho v_r \frac{\partial v_r}{\partial r} = -\frac{\partial p}{\partial r} + \mu \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} (r v_r) \right) + \frac{\partial^2 v_r}{\partial z^2} \right] + \rho g_r$$

### Part (b)

From the continuity equation, the first term in square brackets is zero.

$$\rho v_r \frac{\partial v_r}{\partial r} = -\frac{\partial p}{\partial r} + \mu \frac{\partial^2 v_r}{\partial z^2} + \rho g_r$$

Rewrite the left side and factor a minus sign from the pressure and gravity terms on the right side.

$$\frac{\rho}{2} \frac{\partial}{\partial r} (v_r^2) = - \left( \frac{\partial p}{\partial r} - \rho g_r \right) + \mu \frac{\partial^2 v_r}{\partial z^2}$$

Substitute  $v_r = \phi/r$ .

$$\frac{\rho}{2} \frac{\partial}{\partial r} \left( \frac{\phi^2}{r^2} \right) = -\frac{\partial}{\partial r} (p - \rho g_r r) + \mu \frac{\partial^2}{\partial z^2} \left( \frac{\phi}{r} \right)$$

Introduce the modified pressure function here,

$$\frac{\rho}{2} \phi^2 \frac{\partial}{\partial r} \left( \frac{1}{r^2} \right) = -\frac{d\mathcal{P}}{dr} + \mu \frac{1}{r} \frac{\partial^2 \phi}{\partial z^2},$$

where  $\mathcal{P} = \mathcal{P}(r) = p(r) - \rho g_r r$ . Therefore,

$$-\rho \frac{\phi^2}{r^3} = -\frac{d\mathcal{P}}{dr} + \mu \frac{1}{r} \frac{d^2 \phi}{dz^2}.$$

### Part (c)

If creeping flow is assumed, then the nonlinear term on the left side is zero.

$$0 = -\frac{d\mathcal{P}}{dr} + \mu \frac{1}{r} \frac{d^2 \phi}{dz^2}$$

Bring  $d\mathcal{P}/dr$  to the left side.

$$\frac{d\mathcal{P}}{dr} = \mu \frac{1}{r} \frac{d^2 \phi}{dz^2} \tag{3}$$

Separate variables.

$$d\mathcal{P} = \mu \frac{dr}{r} \frac{d^2 \phi}{dz^2}$$

Integrate both sides.

$$\int_{\mathcal{P}_1}^{\mathcal{P}_2} d\mathcal{P} = \int_{r_1}^{r_2} \mu \frac{dr}{r} \frac{d^2 \phi}{dz^2},$$

where  $\mathcal{P}_1 = \mathcal{P}(r_1)$  and  $\mathcal{P}_2 = \mathcal{P}(r_2)$ .

$$\mathcal{P}_2 - \mathcal{P}_1 = \mu (\ln r) \Big|_{r_1}^{r_2} \frac{d^2 \phi}{dz^2}$$

Plug in the limits.

$$0 = \mathcal{P}_1 - \mathcal{P}_2 + \mu(\ln r_2 - \ln r_1) \frac{d^2 \phi}{dz^2}$$

Therefore,

$$0 = (\mathcal{P}_1 - \mathcal{P}_2) + \left( \mu \ln \frac{r_2}{r_1} \right) \frac{d^2 \phi}{dz^2}.$$

### Part (d)

Solve for the term containing  $\phi$ .

$$\frac{d^2 \phi}{dz^2} = \frac{\mathcal{P}_2 - \mathcal{P}_1}{\mu \ln(r_2/r_1)}$$

Integrate both sides with respect to  $z$ .

$$\frac{d\phi}{dz} = \frac{\mathcal{P}_2 - \mathcal{P}_1}{\mu \ln(r_2/r_1)} z + C_1$$

Integrate both sides with respect to  $z$  once more.

$$\phi(z) = \frac{\mathcal{P}_2 - \mathcal{P}_1}{2\mu \ln(r_2/r_1)} z^2 + C_1 z + C_2$$

Use the boundary conditions for  $v_r$  to obtain those for  $\phi$ .

$$\begin{aligned} v_r(r, -b) = 0 &\quad \rightarrow \quad \frac{\phi(-b)}{r} = 0 &\quad \rightarrow \quad \phi(-b) = 0 \\ v_r(r, b) = 0 &\quad \rightarrow \quad \frac{\phi(b)}{r} = 0 &\quad \rightarrow \quad \phi(b) = 0 \end{aligned}$$

Apply them now to determine  $C_1$  and  $C_2$ .

$$\begin{aligned} \phi(-b) &= \frac{\mathcal{P}_2 - \mathcal{P}_1}{2\mu \ln(r_2/r_1)} b^2 - C_1 b + C_2 = 0 \\ \phi(b) &= \frac{\mathcal{P}_2 - \mathcal{P}_1}{2\mu \ln(r_2/r_1)} b^2 + C_1 b + C_2 = 0 \end{aligned}$$

Solving the system of equations yields

$$C_1 = 0 \quad \text{and} \quad C_2 = -\frac{\mathcal{P}_2 - \mathcal{P}_1}{2\mu \ln(r_2/r_1)} b^2.$$

So then

$$\begin{aligned} \phi(z) &= \frac{\mathcal{P}_2 - \mathcal{P}_1}{2\mu \ln(r_2/r_1)} z^2 - \frac{\mathcal{P}_2 - \mathcal{P}_1}{2\mu \ln(r_2/r_1)} b^2 \\ &= -\frac{\mathcal{P}_2 - \mathcal{P}_1}{2\mu \ln(r_2/r_1)} (b^2 - z^2) \\ &= \frac{(\mathcal{P}_1 - \mathcal{P}_2)b^2}{2\mu \ln(r_2/r_1)} \left[ 1 - \left( \frac{z}{b} \right)^2 \right]. \end{aligned}$$

Therefore, since  $v_r = \phi/r$ ,

$$v_r(r, z) = \frac{(\mathcal{P}_1 - \mathcal{P}_2)b^2}{2\mu r \ln(r_2/r_1)} \left[ 1 - \left( \frac{z}{b} \right)^2 \right].$$

**Part (e)**

The volumetric flow rate is obtained by integrating the velocity distribution over the area the fluid is flowing through.

$$\frac{dV}{dt} = \iint v_r dA$$

Multiply both sides by the density  $\rho$ .

$$\rho \frac{dV}{dt} = \rho \iint v_r dA$$

Because  $\rho$  is constant, it can be brought inside the derivative.

$$\frac{d(\rho V)}{dt} = \rho \iint v_r dA$$

Density times volume is mass.

$$\begin{aligned} \frac{dm}{dt} &= \rho \iint v_r dA \\ &= \rho \int_{-b}^b \int_0^{2\pi} v_r(r, z) (r d\theta) (dz) \\ &= \rho \int_{-b}^b \int_0^{2\pi} \frac{(\mathcal{P}_1 - \mathcal{P}_2)b^2}{2\mu r \ln(r_2/r_1)} \left[ 1 - \left(\frac{z}{b}\right)^2 \right] (r d\theta dz) \\ &= \frac{(\mathcal{P}_1 - \mathcal{P}_2)\rho}{2\mu \ln(r_2/r_1)} \int_{-b}^b \int_0^{2\pi} (b^2 - z^2) d\theta dz \\ &= \frac{(\mathcal{P}_1 - \mathcal{P}_2)\rho}{2\mu \ln(r_2/r_1)} \left( \int_0^{2\pi} d\theta \right) \int_{-b}^b (b^2 - z^2) dz \\ &= \frac{(\mathcal{P}_1 - \mathcal{P}_2)\rho}{2\mu \ln(r_2/r_1)} (2\pi) \left( b^2 z - \frac{z^3}{3} \right) \Big|_{-b}^b \\ &= \frac{(\mathcal{P}_1 - \mathcal{P}_2)\rho}{2\mu \ln(r_2/r_1)} (2\pi) \left( \frac{2b^3}{3} + \frac{2b^3}{3} \right) \end{aligned}$$

Therefore, letting  $w = dm/dt$ ,

$$w = \frac{4\pi(\mathcal{P}_1 - \mathcal{P}_2)b^3\rho}{3\mu \ln(r_2/r_1)}.$$

**Part (f)**

We can solve for the modified pressure function with equation (3).

$$\begin{aligned} \frac{d\mathcal{P}}{dr} &= \frac{\mu}{r} \frac{d^2\phi}{dz^2} \\ &= \frac{1}{r} \frac{\mathcal{P}_2 - \mathcal{P}_1}{\ln(r_2/r_1)} \end{aligned}$$

Integrate both sides with respect to  $r$ .

$$\mathcal{P}(r) = \frac{\mathcal{P}_2 - \mathcal{P}_1}{\ln(r_2/r_1)} \ln r + C_3$$

To determine  $C_3$ , use the condition that  $\mathcal{P}(r_2) = \mathcal{P}_2$ .

$$\mathcal{P}(r_2) = \frac{\mathcal{P}_2 - \mathcal{P}_1}{\ln(r_2/r_1)} \ln r_2 + C_3 = \mathcal{P}_2 \quad \rightarrow \quad C_3 = \frac{\mathcal{P}_1 \ln r_2 - \mathcal{P}_2 \ln r_1}{\ln(r_2/r_1)}$$

So then

$$\begin{aligned} \mathcal{P}(r) &= \frac{\mathcal{P}_2 - \mathcal{P}_1}{\ln(r_2/r_1)} \ln r + \frac{\mathcal{P}_1 \ln r_2 - \mathcal{P}_2 \ln r_1}{\ln(r_2/r_1)} \\ &= \frac{\mathcal{P}_1 \ln \frac{r_2}{r} + \mathcal{P}_2 \ln \frac{r}{r_1}}{\ln \frac{r_2}{r_1}}. \end{aligned}$$

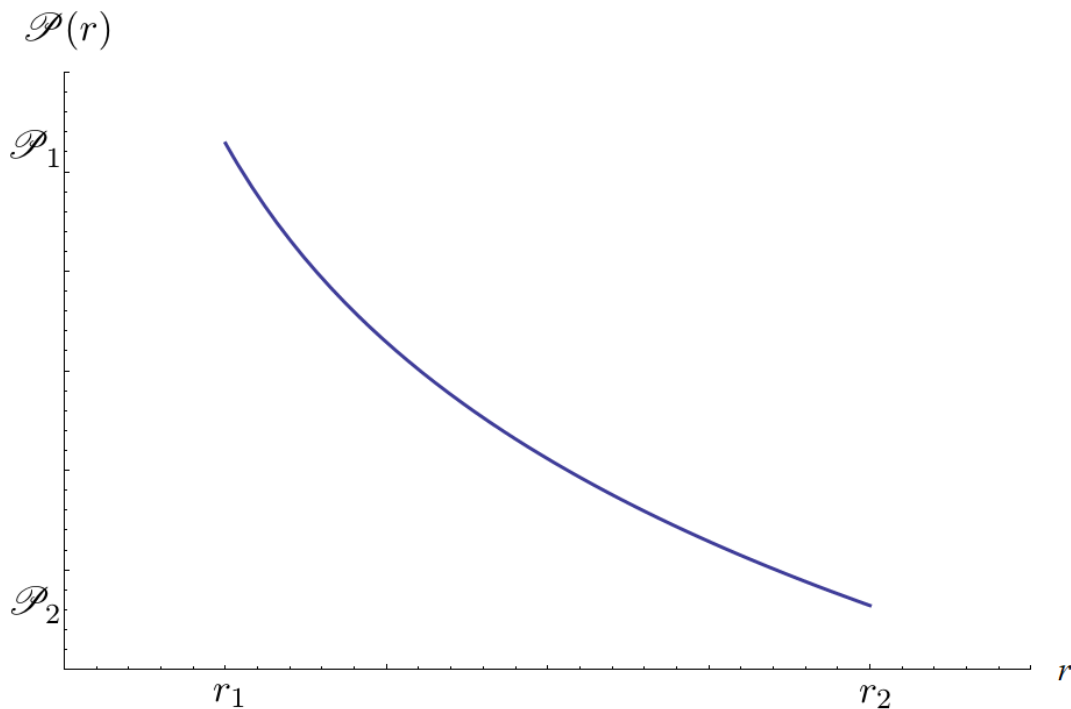


Figure 1: This is a plot of  $\mathcal{P}(r)$  as a function of  $r$ .

Now the velocity field  $\mathbf{v} = v_r(r, z)\hat{\mathbf{r}}$  will be plotted by changing to Cartesian coordinates.

$$\mathbf{v} = v_r(x, y, z)(\cos \theta \hat{\mathbf{x}} + \sin \theta \hat{\mathbf{y}}) = \frac{(\mathcal{P}_1 - \mathcal{P}_2)b^2}{2\mu\sqrt{x^2 + y^2} \ln(r_2/r_1)} \left[ 1 - \left(\frac{z}{b}\right)^2 \right] \left( \frac{x}{\sqrt{x^2 + y^2}} \hat{\mathbf{x}} + \frac{y}{\sqrt{x^2 + y^2}} \hat{\mathbf{y}} \right)$$

If we set

$$\frac{(\mathcal{P}_1 - \mathcal{P}_2)b^2}{2\mu \ln(r_2/r_1)} = 10$$

and  $b = 0.1$ , for example, then we obtain the following velocity field.

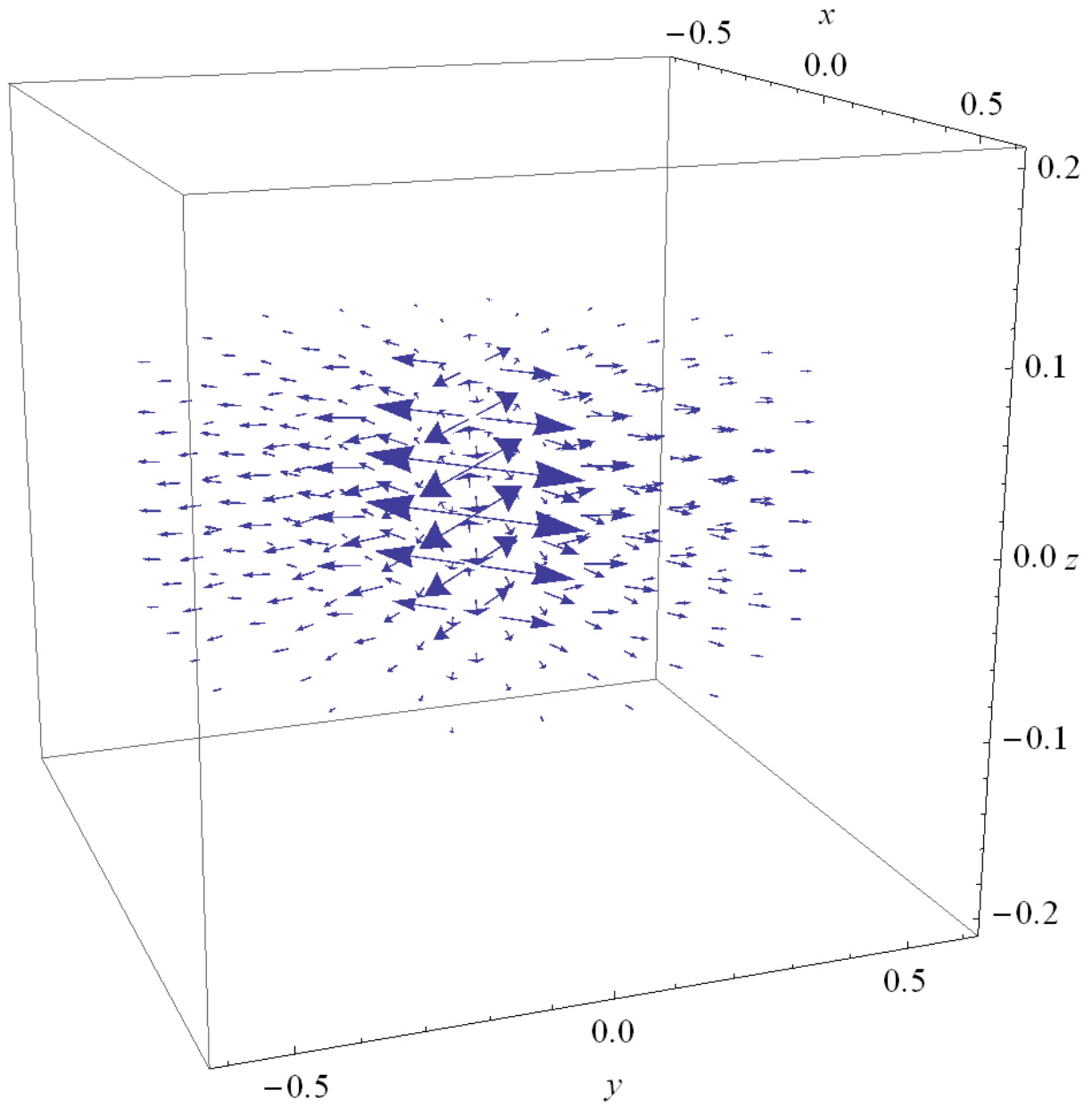


Figure 2: This is a sample plot of the velocity field  $\mathbf{v} = v_r(r, z)\hat{\mathbf{r}}$ .

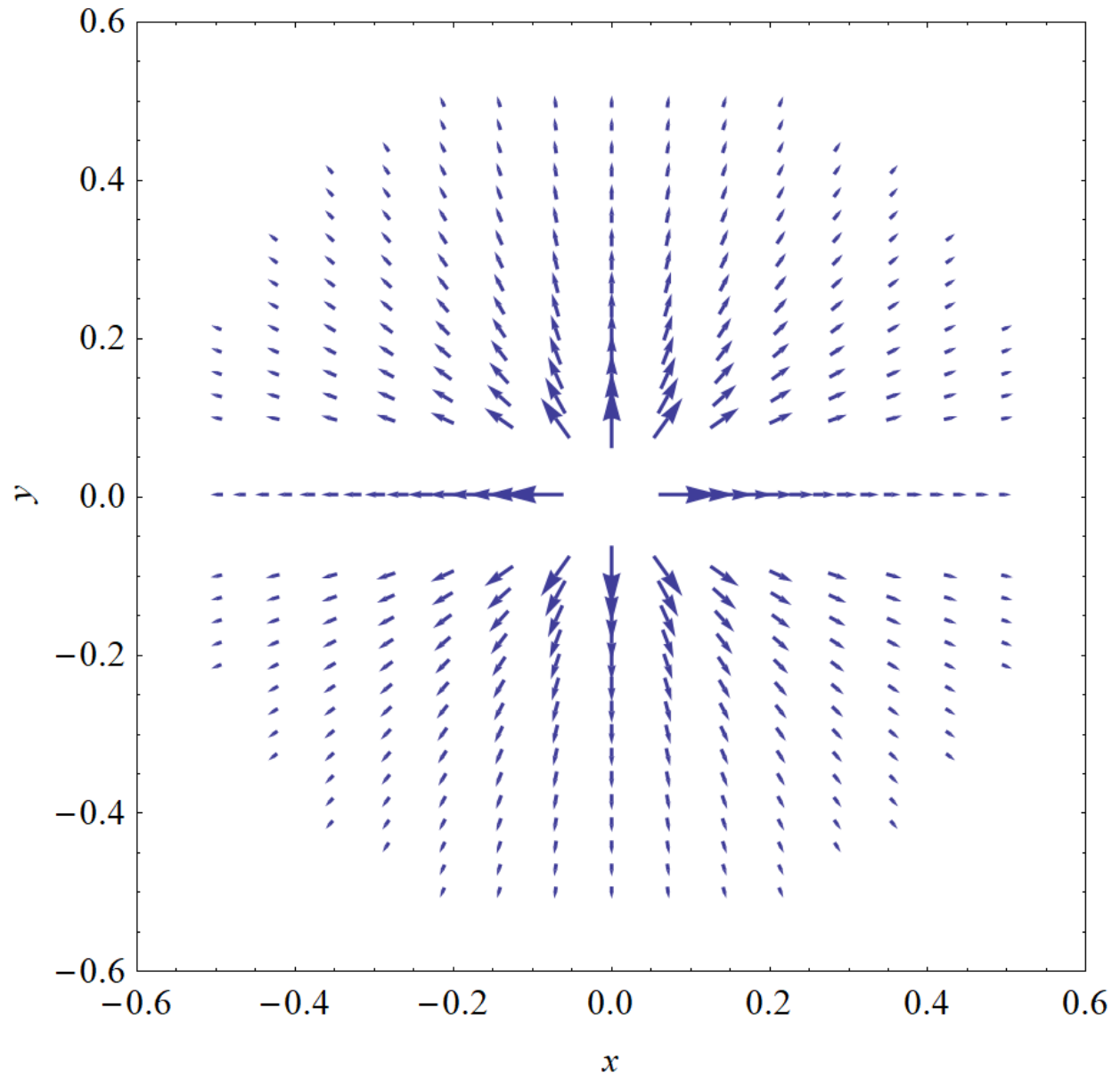


Figure 3: This is a cross-sectional plot of the sample velocity field at  $z = 0$ .