Problem 3B.11

Radial flow between two coaxial cylinders. Consider an incompressible fluid, at constant temperature, flowing radially between two porous cylindrical shells with inner and outer radii κR and R.

(a) Show that the equation of continuity leads to $v_r = C/r$, where C is a constant.

(b) Simplify the components of the equation of motion to obtain the following expressions for the modified-pressure distribution:

$$\frac{d\mathscr{P}}{dr} = -\rho v_r \frac{dv_r}{dr} \quad \frac{d\mathscr{P}}{d\theta} = 0 \quad \frac{d\mathscr{P}}{dz} = 0 \tag{3B.11-1}$$

(c) Integrate the expression for $d\mathscr{P}/dr$ above to get

$$\mathscr{P}(r) - \mathscr{P}(R) = \frac{1}{2}\rho[v_r(R)]^2 \left[1 - \left(\frac{R}{r}\right)^2\right] \tag{3B.11-2}$$

(d) Write out all the nonzero components of $\boldsymbol{\tau}$ for this flow.

(e) Repeat the problem for concentric spheres.

Solution

Part (a)

We assume that the fluid flows only in the r-direction and that the velocity varies as a function of radius only.

$$\mathbf{v} = v_r(r)\hat{\mathbf{r}}$$

The equation of continuity results by considering a mass balance over a volume element that the fluid is flowing through. Because the fluid is assumed to be incompressible, the equation simplifies to

$$\nabla \cdot \mathbf{v} = 0. \tag{1}$$

The equation of motion results by considering a momentum balance over a volume element that the fluid is flowing through. Assuming the fluid viscosity μ is also constant, the equation simplifies to the Navier-Stokes equation.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla \cdot \rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g} \tag{2}$$

As this is a vector equation, it actually represents three scalar equations—one for each variable in the chosen coordinate system. Using cylindrical coordinates is the appropriate choice for this problem, so equations (1) and (2) will be used in (r, θ, z). From Appendix B.4 on page 846, the continuity equation becomes

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \underbrace{\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}}_{=\,0} + \underbrace{\frac{\partial v_z}{\partial z}}_{=\,0} = 0.$$

Multiply both sides by r.

$$\frac{d}{dr}(rv_r) = 0$$

Integrate both sides with respect to r.

$$rv_r = C$$

Therefore, $v_r = C/r$.

Part (b)

From Appendix B.6 on page 848, the Navier-Stokes equation yields the following three scalar equations in cylindrical coordinates.

$$\rho\left(\underbrace{\frac{\partial v_r}{\partial t}}_{=0} + v_r\frac{\partial v_r}{\partial r} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_r}{\partial z}}_{=0} - \underbrace{\frac{v_\theta^2}{r}}_{=0}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right) + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_r}{\partial z^2}}_{=0} - \underbrace{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}}_{=0}\right] + \rho g_r$$

$$\rho\left(\underbrace{\frac{\partial v_\theta}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_\theta}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_\theta}{\partial z}}_{=0} + \underbrace{\frac{v_r v_\theta}{r}}_{=0}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\underbrace{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right)}_{=0} + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_\theta}{\partial z^2}}_{=0} + \underbrace{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}}_{=0}\right] + \rho g_\theta$$

$$\rho\left(\underbrace{\frac{\partial v_z}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_z}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_z}{\partial z}}_{=0}\right) = -\frac{\partial p}{\partial z} + \mu\left[\underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)}_{=0} + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_z}{\partial z^2}}_{=0}\right] + \rho g_z$$

Because rv_r is a constant, the term in square brackets in the first equation is zero. Also, assuming that gravity points downward in the direction of z, we have $g_r = 0$ and $g_\theta = 0$ and $g_z = g$.

$$\rho v_r\frac{\partial v_r}{\partial r} = -\frac{\partial p}{\partial r} \tag{3}$$

$$0 = -\frac{1}{r}\frac{\partial p}{\partial \theta} \tag{4}$$

$$0 = -\frac{\partial p}{\partial z} + \rho g \tag{5}$$

Combine the pressure and gravity terms in equation (5).

$$0 = -\left(\frac{\partial p}{\partial z} - \rho g\right)$$

$$0 = -\frac{\partial}{\partial z}(p - \rho g z)$$

Multiply both sides by -1 and introduce the modified pressure function $\mathscr{P} = \mathscr{P}(r, z) = p(r) - \rho g z$. Since the flow is radial, the modified pressure really only varies with r (that is, z is set to a constant h). The following two boxed equations imply this.

$$\boxed{0 = \frac{\partial \mathscr{P}}{\partial z}}$$

$\rho g z$ can be included in the derivative in equation (4) because it evaluates to zero.

$$0 = -\frac{1}{r}\frac{\partial}{\partial \theta}(p - \rho g z)$$

Multiply both sides by $-r$ and write the equation in terms of $\mathscr{P}$.

$$\boxed{0 = \frac{\partial \mathscr{P}}{\partial \theta}}$$

$\rho g z$ can also be included in the derivative in equation (3) because it evaluates to zero.

$$\rho v_r \frac{\partial v_r}{\partial r} = -\frac{\partial}{\partial r}(p - \rho g z)$$

Multiply both sides by -1 and write the equation in terms of $\mathscr{P}$.

$$-\rho v_r \frac{\partial v_r}{\partial r} = \frac{\partial \mathscr{P}}{\partial r}$$

Ordinary derivatives can be used for both v_r and $\mathscr{P}$ because they vary only as a function of r.

$$\boxed{-\rho v_r \frac{dv_r}{dr} = \frac{d\mathscr{P}}{dr}}$$

Part (c)

Rewrite the left side of the previous equation.

$$-\frac{\rho}{2}\frac{d}{dr}(v_r^2) = \frac{d\mathscr{P}}{dr}$$

Integrate both sides from r to R.

$$-\frac{\rho}{2}(v_r^2)\Big|_r^R = \mathscr{P}\Big|_r^R$$

Insert the limits.

$$-\frac{\rho}{2}[v_r^2(R) - v_r^2(r)] = \mathscr{P}(R) - \mathscr{P}(r)$$

Multiply both sides by -1 and factor $v_r^2(R)$.

$$\frac{\rho}{2}v_r^2(R)\left[1 - \frac{v_r^2(r)}{v_r^2(R)}\right] = \mathscr{P}(r) - \mathscr{P}(R)$$

Substitute $v_r(r) = C/r$ and $v_r(R) = C/R$ in the fraction.

$$\frac{\rho}{2}v_r^2(R)\left(1 - \frac{\frac{C^2}{r^2}}{\frac{C^2}{R^2}}\right) = \mathscr{P}(r) - \mathscr{P}(R)$$

$$\frac{\rho}{2}v_r^2(R)\left(1 - \frac{R^2}{r^2}\right) = \mathscr{P}(r) - \mathscr{P}(R)$$

Therefore,

$$\mathscr{P}(r) - \mathscr{P}(R) = \frac{1}{2}\rho[v_r(R)]^2\left[1 - \left(\frac{R}{r}\right)^2\right].$$

Part (d)

The components of the viscous stress tensor $\boldsymbol{\tau}$ in cylindrical coordinates are listed in Appendix B.1 on page 844. There exist only two nonzero components if the fluid is incompressible and $\mathbf{v} = v_r(r)\hat{\mathbf{r}}$.

$$\tau_{rr} = -\mu\left(2\frac{dv_r}{dr}\right) = -2\mu\frac{d}{dr}\left(\frac{C}{r}\right) = \frac{2\mu C}{r^2}$$

$$\tau_{\theta\theta} = -\mu\left(2\frac{v_r}{r}\right) = -\frac{2\mu}{r}\frac{C}{r} = -\frac{2\mu C}{r^2}$$

Part (e)

Now the radial flow between two porous concentric spheres of radii κR and R will be considered. Equation (1) in spherical coordinates becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \underbrace{\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta)}_{=0} + \underbrace{\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}}_{=0} = 0,$$

where θ is the angle from the polar axis. Multiply both sides by r^2.

$$\frac{d}{dr}(r^2 v_r) = 0$$

Integrate both sides with respect to r.

$$r^2 v_r = C$$

Therefore, $v_r = C/r^2$. Equation (2) yields the following three equations for each component in spherical coordinates.

$$\rho\left(\underbrace{\frac{\partial v_r}{\partial t}}_{=0} + v_r\frac{\partial v_r}{\partial r} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta}}_{=0} + \underbrace{\frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial\phi}}_{=0} - \underbrace{\frac{v_\theta^2 + v_\phi^2}{r}}_{=0}\right) = -\frac{\partial p}{\partial r}$$
$$+ \mu\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 v_r) + \underbrace{\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right)}_{=0} + \underbrace{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2}}_{=0}\right] + \rho g_r$$

$$\rho\left(\underbrace{\frac{\partial v_\theta}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_\theta}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta}}_{=0} + \underbrace{\frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}}_{=0} + \underbrace{\frac{v_r v_\theta - v_\phi^2\cot\theta}{r}}_{=0}\right) = -\frac{1}{r}\frac{\partial p}{\partial\theta}$$
$$+ \mu\left[\underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right)}_{=0} + \underbrace{\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta)\right)}_{=0} + \underbrace{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial\phi^2}}_{=0} + \underbrace{\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}}_{=0} - \underbrace{\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}}_{=0}\right] + \rho g_\theta$$

$$\rho\left(\underbrace{\frac{\partial v_\phi}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_\phi}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_\phi}{\partial\theta}}_{=0} + \underbrace{\frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}}_{=0} + \underbrace{\frac{v_\phi v_r + v_\theta v_\phi\cot\theta}{r}}_{=0}\right) = -\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi}$$
$$+ \mu\left[\underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right)}_{=0} + \underbrace{\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v_\phi\sin\theta)\right)}_{=0} + \underbrace{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial\phi^2}}_{=0} + \underbrace{\frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial\phi}}_{=0} + \underbrace{\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\theta}{\partial\phi}}_{=0}\right] + \rho g_\phi$$

Because $r^2 v_r$ is a constant, the term in square brackets in the first equation is zero. Also, assuming that gravity points downward in the direction of z, we have $\mathbf{g} = g\hat{\mathbf{z}} = g[(\cos\theta)\hat{\mathbf{r}} + (-\sin\theta)\hat{\boldsymbol{\theta}}]$ in spherical coordinates. The components are then $g_r = g\cos\theta$ and $g_\theta = -g\sin\theta$ and $g_\phi = 0$.

$$\rho v_r\frac{\partial v_r}{\partial r} = -\frac{\partial p}{\partial r} + \rho g\cos\theta \tag{6}$$

$$0 = -\frac{1}{r}\frac{\partial p}{\partial\theta} - \rho g\sin\theta \tag{7}$$

$$0 = -\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi} \tag{8}$$

Multiply both sides of equation (7) by $-r$.

$$\begin{aligned} 0 &= \frac{\partial p}{\partial \theta} + \rho g r \sin\theta \\ &= \frac{\partial}{\partial \theta}(p - \rho g r \cos\theta) \end{aligned}$$

Here we introduce the modified pressure function $\mathscr{P} = \mathscr{P}(r,\theta) = p(r) - \rho g r \cos\theta$. Since the flow is radial, the modified pressure really only varies with r (that is, θ is set to a constant θ_0). The following two boxed equations imply this.

$$\boxed{0 = \frac{\partial \mathscr{P}}{\partial \theta}}$$

Multiply both sides of equation (8) by $-r\sin\theta$.

$$0 = \frac{\partial p}{\partial \phi}$$

$\rho g r \cos\theta$ can be included in the derivative because it evaluates to zero.

$$0 = \frac{\partial}{\partial \phi}(p - \rho g r \cos\theta)$$

Write the equation in terms of $\mathscr{P}$.

$$\boxed{0 = \frac{\partial \mathscr{P}}{\partial \phi}}$$

Multiply both sides of equation (6) by -1.

$$\begin{aligned} -\rho v_r \frac{\partial v_r}{\partial r} &= \frac{\partial p}{\partial r} - \rho g \cos\theta \\ &= \frac{\partial}{\partial r}(p - \rho g r \cos\theta) \end{aligned}$$

Therefore,

$$\boxed{-\rho v_r \frac{\partial v_r}{\partial r} = \frac{\partial \mathscr{P}}{\partial r}.}$$

Since v_r and $\mathscr{P}$ only vary as a function of r, ordinary derivatives can be used.

$$-\rho v_r \frac{dv_r}{dr} = \frac{d\mathscr{P}}{dr}$$

Rewrite the left side.

$$-\frac{\rho}{2}\frac{d}{dr}(v_r^2) = \frac{d\mathscr{P}}{dr}$$

Integrate both sides from r to R.

$$-\frac{\rho}{2}(v_r^2)\Big|_r^R = \mathscr{P}\Big|_r^R$$

Insert the limits.

$$-\frac{\rho}{2}[v_r^2(R) - v_r^2(r)] = \mathscr{P}(R) - \mathscr{P}(r)$$

Multiply both sides by -1 and factor $v_r^2(R)$.

$$\frac{\rho}{2}v_r^2(R)\left[1-\frac{v_r^2(r)}{v_r^2(R)}\right]=\mathscr{P}(r)-\mathscr{P}(R)$$

Substitute $v_r(r)=C/r^2$ and $v_r(R)=C/R^2$ in the fraction.

$$\frac{\rho}{2}v_r^2(R)\left(1-\frac{\frac{C^2}{r^4}}{\frac{C^2}{R^4}}\right)=\mathscr{P}(r)-\mathscr{P}(R)$$

$$\frac{\rho}{2}v_r^2(R)\left(1-\frac{R^4}{r^4}\right)=\mathscr{P}(r)-\mathscr{P}(R)$$

Therefore,

$$\boxed{\mathscr{P}(r)-\mathscr{P}(R)=\frac{1}{2}\rho[v_r(R)]^2\left[1-\left(\frac{R}{r}\right)^4\right].}$$

The components of the viscous stress tensor $\boldsymbol{\tau}$ in spherical coordinates are listed in Appendix B.1 on page 844. There exist only three nonzero components if the fluid is incompressible and $\mathbf{v}=v_r(r)\hat{\mathbf{r}}$.

$$\begin{aligned}
\tau_{rr}&=-\mu\left(2\frac{dv_r}{dr}\right)=-2\mu\frac{d}{dr}\left(\frac{C}{r^2}\right)=\frac{4\mu C}{r^3}\\
\tau_{\theta\theta}&=-\mu\left(2\frac{v_r}{r}\right)=-\frac{2\mu}{r}\frac{C}{r^2}=-\frac{2\mu C}{r^3}\\
\tau_{\phi\phi}&=-\mu\left(2\frac{v_r}{r}\right)=-\frac{2\mu}{r}\frac{C}{r^2}=-\frac{2\mu C}{r^3}
\end{aligned}$$