Problem 3B.16

Flow in a slit with uniform cross flow (Fig. 3B.16). A fluid flows in the positive x-direction through a long flat duct of length L, width W, and thickness B, where $L \gg W \gg B$. The duct has porous walls at $y = 0$ and $y = B$, so that a constant cross flow can be maintained, with $v_y = v_0$, a constant, everywhere. Flows of this type are important in connection with separation processes using the sweep-diffusion effect. By carefully controlling the cross flow, one can concentrate the larger constituents (molecules, dust particles, etc.) near the upper wall.

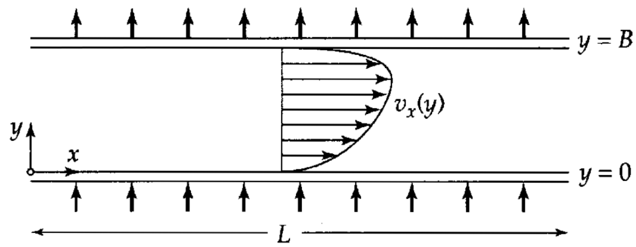

Fig. 3B.16. Flow in a slit of length L, width W, and thickness B. The walls at $y = 0$ and $y = B$ are porous, and there is a flow of the fluid in the y direction, with a uniform velocity $v_y = v_0$.

(a) Show that the velocity profile for the system is given by

$$v_x = \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2}{\mu L}\frac{1}{A}\left(\frac{y}{B} - \frac{e^{Ay/B} - 1}{e^A - 1}\right) \tag{3B.16-1}$$

in which $A = Bv_0\rho/\mu$.

(b) Show that the mass flow rate in the x direction is

$$w = \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^3W\rho}{\mu L}\frac{1}{A}\left(\frac{1}{2} - \frac{1}{A} + \frac{1}{e^A - 1}\right) \tag{3B.16-2}$$

(c) Verify that the above results simplify to those of Problem 2B.3 in the limit that there is no cross flow at all (that is, $A \to 0$).

(d) A colleague has also solved this problem, but taking a coordinate system with $y = 0$ at the midplane of the slit, with the porous walls located at $y = \pm b$. His answer to part (a) above is

$$\frac{v_x}{\langle v_x \rangle} = \frac{e^{\alpha\eta} - \eta\sinh\alpha - \cosh\alpha}{(1/\alpha)\sinh\alpha - \cosh\alpha} \tag{3B.16-3}$$

in which $\alpha = bv_0\rho/\mu$ and $\eta = y/b$. Is this result equivalent to Eq. 3B.16-1?

Solution

Part (a)

We assume that the fluid flows in the x-direction and that the velocity varies as a function of y. In addition, because of the cross flow, there is a constant velocity v_0 in the y-direction.

$$\mathbf{v} = v_x(y)\hat{\boldsymbol{x}} + v_0\hat{\boldsymbol{y}}$$

If we assume the fluid does not slip on the walls, then it has the wall's velocity at $y = 0$ and $y = B$.

$$\begin{aligned} &\text{Boundary Condition 1:} && v_x(0) = 0 \\ &\text{Boundary Condition 2:} && v_x(B) = 0 \end{aligned}$$

The equation of continuity results by considering a mass balance over a volume element that the fluid is flowing through. Assuming the fluid density ρ is constant, the equation simplifies to

$$\nabla \cdot \mathbf{v} = 0. \tag{1}$$

The equation of motion results by considering a momentum balance over a volume element that the fluid is flowing through. Assuming the fluid viscosity μ is also constant, the equation simplifies to the Navier-Stokes equation.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla \cdot \rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g} \tag{2}$$

As this is a vector equation, it actually represents three scalar equations—one for each variable in the chosen coordinate system. Using Cartesian coordinates is the appropriate choice for this problem, so equations (1) and (2) will be used in (x, y, z). From Appendix B.4 on page 846, the continuity equation becomes

$$\underbrace{\frac{\partial v_x}{\partial x}}_{=0} + \underbrace{\frac{\partial v_y}{\partial y}}_{=0} + \underbrace{\frac{\partial v_z}{\partial z}}_{=0} = 0,$$

which doesn't tell us anything. From Appendix B.6 on page 848, the Navier-Stokes equation yields the following three scalar equations in Cartesian coordinates.

$$\begin{aligned}
\rho\left(\underbrace{\frac{\partial v_x}{\partial t}}_{=0} + \underbrace{v_x\frac{\partial v_x}{\partial x}}_{=0} + v_y\frac{\partial v_x}{\partial y} + \underbrace{v_z\frac{\partial v_x}{\partial z}}_{=0}\right) &= -\frac{\partial p}{\partial x} + \mu\left[\underbrace{\frac{\partial^2 v_x}{\partial x^2}}_{=0} + \frac{\partial^2 v_x}{\partial y^2} + \underbrace{\frac{\partial^2 v_x}{\partial z^2}}_{=0}\right] + \rho g_x \\
\rho\left(\underbrace{\frac{\partial v_y}{\partial t}}_{=0} + \underbrace{v_x\frac{\partial v_y}{\partial x}}_{=0} + \underbrace{v_y\frac{\partial v_y}{\partial y}}_{=0} + \underbrace{v_z\frac{\partial v_y}{\partial z}}_{=0}\right) &= -\frac{\partial p}{\partial y} + \mu\left[\underbrace{\frac{\partial^2 v_y}{\partial x^2}}_{=0} + \underbrace{\frac{\partial^2 v_y}{\partial y^2}}_{=0} + \underbrace{\frac{\partial^2 v_y}{\partial z^2}}_{=0}\right] + \rho g_y \\
\rho\left(\underbrace{\frac{\partial v_z}{\partial t}}_{=0} + \underbrace{v_x\frac{\partial v_z}{\partial x}}_{=0} + \underbrace{v_y\frac{\partial v_z}{\partial y}}_{=0} + \underbrace{v_z\frac{\partial v_z}{\partial z}}_{=0}\right) &= -\frac{\partial p}{\partial z} + \mu\left[\underbrace{\frac{\partial^2 v_z}{\partial x^2}}_{=0} + \underbrace{\frac{\partial^2 v_z}{\partial y^2}}_{=0} + \underbrace{\frac{\partial^2 v_z}{\partial z^2}}_{=0}\right] + \rho g_z
\end{aligned}$$

The relevant equation for the velocity is the x-equation, which has simplified considerably from the assumption that $\mathbf{v} = v_x(y)\hat{\boldsymbol{x}} + v_0\hat{\boldsymbol{y}}$.

$$\rho v_0\frac{dv_x}{dy} = -\frac{\partial p}{\partial x} + \mu\frac{d^2 v_x}{dy^2} + \rho g_x$$

The sum of $-\partial p/\partial x$ and ρg_x is the modified pressure gradient across the duct.

$$\rho v_0 \frac{dv_x}{dy} = -\frac{(\mathscr{P}_L - \mathscr{P}_0)}{L - 0} + \mu \frac{d^2 v_x}{dy^2}$$

The velocity distribution thus satisfies the following equation.

$$\frac{d^2 v_x}{dy^2} - \frac{\rho v_0}{\mu}\frac{dv_x}{dy} = \frac{\mathscr{P}_L - \mathscr{P}_0}{L} \tag{3}$$

This ODE is first-order in dv_x/dy, so we can use an integrating factor I to solve it.

$$I = \exp\left(\int^y -\frac{\rho v_0}{\mu}\,ds\right) = e^{-\rho v_0 y/\mu}$$

Multiply both sides of the ODE by I.

$$e^{-\rho v_0 y/\mu}\frac{d^2 v_x}{dy^2} - \frac{\rho v_0}{\mu} e^{-\rho v_0 y/\mu}\frac{dv_x}{dy} = \frac{\mathscr{P}_L - \mathscr{P}_0}{L} e^{-\rho v_0 y/\mu}$$

The left side can be written as $d/dy(Iv_x')$ by the product rule.

$$\frac{d}{dy}\left(e^{-\rho v_0 y/\mu}\frac{dv_x}{dy}\right) = \frac{\mathscr{P}_L - \mathscr{P}_0}{L} e^{-\rho v_0 y/\mu}$$

Integrate both sides with respect to y.

$$e^{-\rho v_0 y/\mu}\frac{dv_x}{dy} = \frac{\mathscr{P}_L - \mathscr{P}_0}{L}\left(-\frac{\mu}{\rho v_0}\right) e^{-\rho v_0 y/\mu} + C_1$$

Multiply both sides by $e^{\rho v_0 y/\mu}$.

$$\frac{dv_x}{dy} = \frac{\mathscr{P}_L - \mathscr{P}_0}{L}\left(-\frac{\mu}{\rho v_0}\right) + C_1 e^{\rho v_0 y/\mu}$$

Integrate both sides with respect to y once more.

$$v_x(y) = \frac{\mathscr{P}_L - \mathscr{P}_0}{L}\left(-\frac{\mu}{\rho v_0}\right) y + C_1\left(\frac{\mu}{\rho v_0}\right) e^{\rho v_0 y/\mu} + C_2$$

Now apply the two boundary conditions to determine C_1 and C_2.

$$\begin{aligned} v_x(0) &= C_1\left(\frac{\mu}{\rho v_0}\right) + C_2 = 0 \\ v_x(B) &= \frac{\mathscr{P}_L - \mathscr{P}_0}{L}\left(-\frac{\mu}{\rho v_0}\right) B + C_1\left(\frac{\mu}{\rho v_0}\right) e^{\rho v_0 B/\mu} + C_2 = 0 \end{aligned}$$

Solving this system of equations yields

$$C_1 = \frac{B}{L}\frac{\mathscr{P}_0 - \mathscr{P}_L}{1 - e^{\rho v_0 B/\mu}} \quad \text{and} \quad C_2 = -\frac{\mu B}{\rho v_0 L}\frac{\mathscr{P}_0 - \mathscr{P}_L}{1 - e^{\rho v_0 B/\mu}}.$$

Plug these constants into the general solution and simplify it until the desired result is obtained.

$$\begin{aligned}
v_x(y) &= \frac{\mathscr{P}_L - \mathscr{P}_0}{L}\left(-\frac{\mu}{\rho v_0}\right) y + \left(\frac{B}{L}\frac{\mathscr{P}_0 - \mathscr{P}_L}{1 - e^{\rho v_0 B/\mu}}\right)\left(\frac{\mu}{\rho v_0}\right) e^{\rho v_0 y/\mu} + \left(-\frac{\mu B}{\rho v_0 L}\frac{\mathscr{P}_0 - \mathscr{P}_L}{1 - e^{\rho v_0 B/\mu}}\right) \\
&= \frac{\mu B}{\rho v_0 L}(\mathscr{P}_0 - \mathscr{P}_L)\left(\frac{y}{B} + \frac{1}{1 - e^{\rho v_0 B/\mu}} e^{\rho v_0 y/\mu} - \frac{1}{1 - e^{\rho v_0 B/\mu}}\right) \\
&= \frac{\mu B}{\rho v_0 L}(\mathscr{P}_0 - \mathscr{P}_L)\left(\frac{y}{B} + \frac{e^{\rho v_0 y/\mu} - 1}{1 - e^{\rho v_0 B/\mu}}\right) \\
&= \frac{\mu B}{\rho v_0 L}(\mathscr{P}_0 - \mathscr{P}_L)\left(\frac{y}{B} - \frac{e^{\rho v_0 y/\mu} - 1}{e^{\rho v_0 B/\mu} - 1}\right)
\end{aligned}$$

Therefore, letting $A = \rho v_0 B/\mu$,

$$v_x(y) = \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2}{\mu L}\frac{1}{A}\left(\frac{y}{B} - \frac{e^{Ay/B} - 1}{e^A - 1}\right).$$

Part (b)

The volumetric flow rate through the slit is given by the integral of the velocity distribution over the cross-sectional area that the fluid flows through.

$$\frac{dV}{dt} = \int v_x \, dA$$

To get the mass flow rate, multiply both sides by the mass density ρ.

$$\rho \frac{dV}{dt} = \rho \int v_x \, dA$$

$$\frac{d(\rho V)}{dt} = \rho \int v_x \, dA$$

Use the fact that mass is density times volume and evaluate the right side.

$$\begin{aligned}
\frac{dm}{dt} &= \rho \int v_x \, dA \\
&= \rho \int_0^B v_x(y)(W\, dy) \\
&= W\rho \int_0^B v_x(y)\, dy \\
&= W\rho \int_0^B \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2}{\mu L}\frac{1}{A}\left(\frac{y}{B} - \frac{e^{Ay/B} - 1}{e^A - 1}\right) dy \\
&= W\rho \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2}{\mu L}\frac{1}{A}\left[\frac{1}{B}\int_0^B y\, dy - \frac{1}{e^A - 1}\int_0^B (e^{Ay/B} - 1)\, dy\right]
\end{aligned}$$

Calculate these integrals.

$$\begin{aligned}
\frac{dm}{dt} &= W\rho\frac{(\mathscr{P}_0-\mathscr{P}_L)B^2}{\mu L}\frac{1}{A}\left[\frac{1}{B}\left(\frac{B^2}{2}\right)-\frac{1}{e^A-1}\left(\frac{B}{A}e^{Ay/B}-y\right)\Big|_0^B\right] \\
&= W\rho\frac{(\mathscr{P}_0-\mathscr{P}_L)B^2}{\mu L}\frac{1}{A}\left[\frac{B}{2}-\frac{1}{e^A-1}\left(\frac{B}{A}e^A-\frac{B}{A}-B\right)\right] \\
&= W\rho\frac{(\mathscr{P}_0-\mathscr{P}_L)B^2}{\mu L}\frac{1}{A}\left[\frac{B}{2}-\frac{1}{e^A-1}\frac{B}{A}(e^A-1)+\frac{B}{e^A-1}\right] \\
&= \frac{(\mathscr{P}_0-\mathscr{P}_L)B^2W\rho}{\mu L}\frac{1}{A}\left(\frac{B}{2}-\frac{B}{A}+\frac{B}{e^A-1}\right)
\end{aligned}$$

Therefore, the mass flow rate in the slit is

$$w=\frac{dm}{dt}=\frac{(\mathscr{P}_0-\mathscr{P}_L)B^3W\rho}{\mu L}\frac{1}{A}\left(\frac{1}{2}-\frac{1}{A}+\frac{1}{e^A-1}\right).$$

Part (c)

First check that the velocity distribution reduces to the corresponding result in Problem 2B.3. Take the limit of $v_x(y)$ as $A\to 0$, using l'Hôpital's rule whenever a 0/0 indeterminate form appears.

$$\begin{aligned}
\lim_{A\to 0} v_x(y) &= \lim_{A\to 0}\frac{(\mathscr{P}_0-\mathscr{P}_L)B^2}{\mu L}\frac{1}{A}\left(\frac{y}{B}-\frac{e^{Ay/B}-1}{e^A-1}\right) \\
&= \frac{(\mathscr{P}_0-\mathscr{P}_L)B^2}{\mu L}\lim_{A\to 0}\frac{1}{A}\left(\frac{y}{B}-\frac{e^{Ay/B}-1}{e^A-1}\right) \\
&= \frac{(\mathscr{P}_0-\mathscr{P}_L)B^2}{\mu L}\lim_{A\to 0}\frac{1}{A}\frac{y(e^A-1)-B(e^{Ay/B}-1)}{B(e^A-1)} \\
&= \frac{(\mathscr{P}_0-\mathscr{P}_L)B^2}{\mu L}\lim_{A\to 0}\frac{ye^A-y-Be^{Ay/B}+B}{BAe^A-BA} \\
&\overset{\frac{0}{0}}{\underset{\text{H}}{=}} \frac{(\mathscr{P}_0-\mathscr{P}_L)B^2}{\mu L}\lim_{A\to 0}\frac{\frac{\partial}{\partial A}(ye^A-y-Be^{Ay/B}+B)}{\frac{\partial}{\partial A}(BAe^A-BA)} \\
&= \frac{(\mathscr{P}_0-\mathscr{P}_L)B^2}{\mu L}\lim_{A\to 0}\frac{ye^A-ye^{Ay/B}}{Be^A+BAe^A-B} \\
&\overset{\frac{0}{0}}{\underset{\text{H}}{=}} \frac{(\mathscr{P}_0-\mathscr{P}_L)B^2}{\mu L}\lim_{A\to 0}\frac{\frac{\partial}{\partial A}(ye^A-ye^{Ay/B})}{\frac{\partial}{\partial A}(Be^A+BAe^A-B)} \\
&= \frac{(\mathscr{P}_0-\mathscr{P}_L)B^2}{\mu L}\lim_{A\to 0}\frac{ye^A-\frac{y^2}{B}e^{Ay/B}}{Be^A+Be^A+BAe^A} \\
&= \frac{(\mathscr{P}_0-\mathscr{P}_L)B^2}{\mu L}\left(\frac{y-\frac{y^2}{B}}{2B}\right) \\
&= \frac{(\mathscr{P}_0-\mathscr{P}_L)B^2}{2\mu L}\left(\frac{y}{B}-\frac{y^2}{B^2}\right)
\end{aligned}$$

In order to get the result from Problem 2B.3, replace B with $2B$ and replace y with $y+B$. The first replacement widens the gap of the slit by two times, and the second replacement makes it so that the origin is at the midline.

$$\begin{aligned}
\lim_{A\to 0} v_x(y) &= \frac{(\mathscr{P}_0 - \mathscr{P}_L)(2B)^2}{2\mu L}\left[\frac{(y+B)}{(2B)} - \frac{(y+B)^2}{(2B)^2}\right] \\
&= \frac{\mathscr{P}_0 - \mathscr{P}_L}{2\mu L}[2B(y+B) - (y+B)^2] \\
&= \frac{\mathscr{P}_0 - \mathscr{P}_L}{2\mu L}(B^2 - y^2) \\
&= \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2}{2\mu L}\left(1 - \frac{y^2}{B^2}\right) \\
&= \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2}{2\mu L}\left[1 - \left(\frac{y}{B}\right)^2\right]
\end{aligned}$$

Secondly, check that the mass flow rate reduces to the corresponding result in Problem 2B.3. Take the limit of w as $A \to 0$, using l'Hôpital's rule whenever a $0/0$ indeterminate form appears.

$$\begin{aligned}
\lim_{A\to 0} w &= \lim_{A\to 0} \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^3 W\rho}{\mu L}\frac{1}{A}\left(\frac{1}{2} - \frac{1}{A} + \frac{1}{e^A - 1}\right) \\
&= \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^3 W\rho}{\mu L}\lim_{A\to 0}\frac{1}{A}\left(\frac{1}{2} - \frac{1}{A} + \frac{1}{e^A - 1}\right) \\
&= \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^3 W\rho}{\mu L}\lim_{A\to 0}\frac{1}{A}\left(\frac{A-2}{2A} + \frac{1}{e^A - 1}\right) \\
&= \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^3 W\rho}{\mu L}\lim_{A\to 0}\frac{1}{A}\frac{(A-2)(e^A-1) + 2A(1)}{2A(e^A - 1)} \\
&= \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^3 W\rho}{\mu L}\lim_{A\to 0}\frac{Ae^A - 2e^A + 2 + A}{2A^2e^A - 2A^2} \\
&\overset{\frac{0}{0}}{\underset{\text{H}}{=}} \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^3 W\rho}{\mu L}\lim_{A\to 0}\frac{\frac{d}{dA}(Ae^A - 2e^A + 2 + A)}{\frac{d}{dA}(2A^2e^A - 2A^2)} \\
&= \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^3 W\rho}{\mu L}\lim_{A\to 0}\frac{Ae^A - e^A + 1}{4Ae^A + 2A^2e^A - 4A} \\
&\overset{\frac{0}{0}}{\underset{\text{H}}{=}} \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^3 W\rho}{\mu L}\lim_{A\to 0}\frac{\frac{d}{dA}(Ae^A - e^A + 1)}{\frac{d}{dA}(4Ae^A + 2A^2e^A - 4A)} \\
&= \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^3 W\rho}{\mu L}\lim_{A\to 0}\frac{Ae^A}{4e^A + 8Ae^A + 2A^2e^A - 4} \\
&\overset{\frac{0}{0}}{\underset{\text{H}}{=}} \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^3 W\rho}{\mu L}\lim_{A\to 0}\frac{\frac{d}{dA}(Ae^A)}{\frac{d}{dA}(4e^A + 8Ae^A + 2A^2e^A - 4)} \\
&= \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^3 W\rho}{\mu L}\lim_{A\to 0}\frac{e^A + Ae^A}{4e^A + 8e^A + 8Ae^A + 4Ae^A + 2A^2e^A} \\
&= \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^3 W\rho}{\mu L}\left(\frac{1}{12}\right) \\
&= \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^3 W\rho}{12\mu L}
\end{aligned}$$

To get the result of Problem 2B.3, replace B with $2B$.

$$\lim_{A\to 0} w = \frac{(\mathscr{P}_0 - \mathscr{P}_L)(2B)^3 W\rho}{12\mu L} = \frac{2}{3}\frac{(\mathscr{P}_0 - \mathscr{P}_L)B^3 W\rho}{\mu L}$$

Part (d)

Start with the result of part (a).

$$v_x(y) = \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2}{\mu L}\frac{1}{A}\left(\frac{y}{B} - \frac{e^{Ay/B} - 1}{e^A - 1}\right).$$

Use it to calculate the average velocity in the slit.

$$\begin{aligned}
\langle v_x \rangle = \frac{\displaystyle\int v_x\, dA}{\displaystyle\int dA} = \frac{\displaystyle\int_0^B v_x(y)(W\,dy)}{\displaystyle\int_0^B W\,dy} &= \frac{1}{B}\int_0^B v_x(y)\,dy \\
&= \frac{1}{B}\int_0^B \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2}{\mu L}\frac{1}{A}\left(\frac{y}{B} - \frac{e^{Ay/B} - 1}{e^A - 1}\right) dy \\
&= \frac{(\mathscr{P}_0 - \mathscr{P}_L)B}{\mu L}\frac{1}{A}\left[\frac{1}{B}\int_0^B y\,dy - \frac{1}{e^A - 1}\int_0^B (e^{Ay/B} - 1)\,dy\right] \\
&= \frac{(\mathscr{P}_0 - \mathscr{P}_L)B}{\mu L}\frac{1}{A}\left[\frac{1}{B}\left(\frac{B^2}{2}\right) - \frac{1}{e^A - 1}\left(\frac{B}{A}e^{Ay/B} - y\right)\bigg|_0^B\right] \\
&= \frac{(\mathscr{P}_0 - \mathscr{P}_L)B}{\mu L}\frac{1}{A}\left[\frac{B}{2} - \frac{1}{e^A - 1}\left(\frac{B}{A}e^A - \frac{B}{A} - B\right)\right] \\
&= \frac{(\mathscr{P}_0 - \mathscr{P}_L)B}{\mu L}\frac{1}{A}\left[\frac{B}{2} - \frac{1}{e^A - 1}\frac{B}{A}(e^A - 1) + \frac{B}{e^A - 1}\right] \\
&= \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2}{\mu L}\frac{1}{A}\left(\frac{1}{2} - \frac{1}{A} + \frac{1}{e^A - 1}\right)
\end{aligned}$$

Divide v_x by $\langle v_x \rangle$ and try to get the right side that the colleague found.

$$\begin{aligned}
\frac{v_x}{\langle v_x \rangle} &= \frac{\frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2}{\mu L}\frac{1}{A}\left(\frac{y}{B} - \frac{e^{Ay/B} - 1}{e^A - 1}\right)}{\frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2}{\mu L}\frac{1}{A}\left(\frac{1}{2} - \frac{1}{A} + \frac{1}{e^A - 1}\right)} \\
&= \frac{\frac{y}{B} - \frac{e^{Ay/B} - 1}{e^A - 1}}{\frac{1}{2} - \frac{1}{A} + \frac{1}{e^A - 1}} \\
&= \frac{\frac{y(e^A - 1) - B(e^{Ay/B} - 1)}{B(e^A - 1)}}{\frac{A-2}{2A} + \frac{1}{e^A - 1}} \\
&= \frac{\frac{y(e^A - 1) - B(e^{Ay/B} - 1)}{B(e^A - 1)}}{\frac{(A-2)(e^A - 1) + 2A}{2A(e^A - 1)}} \\
&= \frac{2A}{B}\frac{y(e^A - 1) - B(e^{Ay/B} - 1)}{(A-2)(e^A - 1) + 2A}
\end{aligned}$$

In order to match the colleague's coordinate system, replace B with $2b$ and replace y with $y + b$. This first replacement makes it so that the distance from the midline to the wall is $b = B/2$. The second replacement makes it so that the origin moves up a distance b to the midline.

$$\frac{v_x}{\langle v_x \rangle} = \frac{A}{b} \frac{(y+b)(e^A - 1) - 2b\left[e^{(A/2b)(y+b)} - 1\right]}{(A-2)(e^A - 1) + 2A}$$

Now match the colleague's notation by noticing that

$$A = \frac{\rho v_0 B}{\mu} = \frac{\rho v_0 (2b)}{\mu} = 2\frac{\rho v_0 b}{\mu} = 2\alpha.$$

Making this substitution results in

$$\begin{aligned}
\frac{v_x}{\langle v_x \rangle} &= \frac{2\alpha}{b} \frac{(y+b)(e^{2\alpha} - 1) - 2b\left[e^{(\alpha/b)(y+b)} - 1\right]}{(2\alpha-2)(e^{2\alpha} - 1) + 2(2\alpha)} \\
&= \alpha \frac{\left(\frac{y}{b} + 1\right)(e^{2\alpha} - 1) - 2\left[e^{(\alpha/b)(y+b)} - 1\right]}{(\alpha-1)(e^{2\alpha} - 1) + 2\alpha} \\
&= \alpha \frac{\left(\frac{y}{b} + 1\right)(e^{2\alpha} - 1) - 2(e^{\alpha y/b + \alpha} - 1)}{(\alpha-1)(e^{2\alpha} - 1) + 2\alpha}.
\end{aligned}$$

Use the colleague's last variable $\eta = y/b$.

$$\begin{aligned}
\frac{v_x}{\langle v_x \rangle} &= \alpha \frac{(\eta+1)(e^{2\alpha} - 1) - 2(e^{\alpha\eta}e^{\alpha} - 1)}{(\alpha-1)(e^{2\alpha} - 1) + 2\alpha} \\
&= \alpha \frac{e^{\alpha}}{e^{\alpha}} \frac{(\eta+1)(e^{\alpha} - e^{-\alpha}) - 2(e^{\alpha\eta} - e^{-\alpha})}{(\alpha-1)(e^{\alpha} - e^{-\alpha}) + 2\alpha e^{-\alpha}} \\
&= \alpha \frac{\eta(e^{\alpha} - e^{-\alpha}) + e^{\alpha} + e^{-\alpha} - 2e^{\alpha\eta}}{\alpha(e^{\alpha} + e^{-\alpha}) - (e^{\alpha} - e^{-\alpha})} \\
&= \alpha \frac{\eta(2\sinh\alpha) + (2\cosh\alpha) - 2e^{\alpha\eta}}{\alpha(2\cosh\alpha) - (2\sinh\alpha)} \\
&= \alpha \frac{\eta\sinh\alpha + \cosh\alpha - e^{\alpha\eta}}{\alpha\cosh\alpha - \sinh\alpha} \\
&= \alpha \frac{e^{\alpha\eta} - \eta\sinh\alpha - \cosh\alpha}{\sinh\alpha - \alpha\cosh\alpha} \\
&= \frac{e^{\alpha\eta} - \eta\sinh\alpha - \cosh\alpha}{\frac{1}{\alpha}(\sinh\alpha - \alpha\cosh\alpha)}
\end{aligned}$$

Therefore, using the colleague's coordinate system and notation,

$$\frac{v_x}{\langle v_x \rangle} = \frac{e^{\alpha\eta} - \eta\sinh\alpha - \cosh\alpha}{(1/\alpha)\sinh\alpha - \cosh\alpha},$$

where $\alpha = \rho v_0 b/\mu$ and $\eta = y/b$. This result is equivalent to the result of part (a) because it also satisfies equation (3), the ODE for the velocity distribution, but with $v_x/\langle v_x \rangle = 0$ at $y = \pm b$ for the boundary conditions instead.

$$\begin{aligned}
\frac{d^2 v_x}{dy^2} - \frac{\rho v_0}{\mu}\frac{dv_x}{dy} = \frac{\mathscr{P}_L - \mathscr{P}_0}{L} \quad &\rightarrow \quad \frac{1}{\langle v_x \rangle}\frac{d^2 v_x}{dy^2} - \frac{\rho v_0}{\mu}\frac{1}{\langle v_x \rangle}\frac{dv_x}{dy} = \frac{\mathscr{P}_L - \mathscr{P}_0}{L}\frac{1}{\langle v_x \rangle} \\
&\rightarrow \quad \frac{d^2}{dy^2}\left(\frac{v_x}{\langle v_x \rangle}\right) - \frac{\rho v_0}{\mu}\frac{d}{dy}\left(\frac{v_x}{\langle v_x \rangle}\right) = \frac{\mathscr{P}_L - \mathscr{P}_0}{L}\frac{1}{\langle v_x \rangle}
\end{aligned}$$