Problem 3B.2

Laminar flow in a triangular duct (Fig. 3B.2).[2] One type of compact heat exchanger is shown in Fig. 3B.2(*a*). In order to analyze the performance of such an apparatus, it is necessary to understand the flow in a duct whose cross section is an equilateral triangle. This is done most easily by installing a coordinate system as shown in Fig. 3B.2(*b*).

(a) Verify that the velocity distribution for the laminar flow of a Newtonian fluid in a duct of this type is given by

$$v_z = \frac{(\mathscr{P}_0 - \mathscr{P}_L)}{4\mu LH}(y - H)(3x^2 - y^2) \tag{3B.2-1}$$

(b) From Eq. 3B.2-1 find the average velocity, maximum velocity, and mass flow rate.

Answers: **(b)** $\langle v_z \rangle = \dfrac{(\mathscr{P}_0 - \mathscr{P}_L)H^2}{60\mu L} = \dfrac{9}{20} v_{z,\max}$;

$$w = \frac{\sqrt{3}(\mathscr{P}_0 - \mathscr{P}_L)H^4\rho}{180\mu L}$$

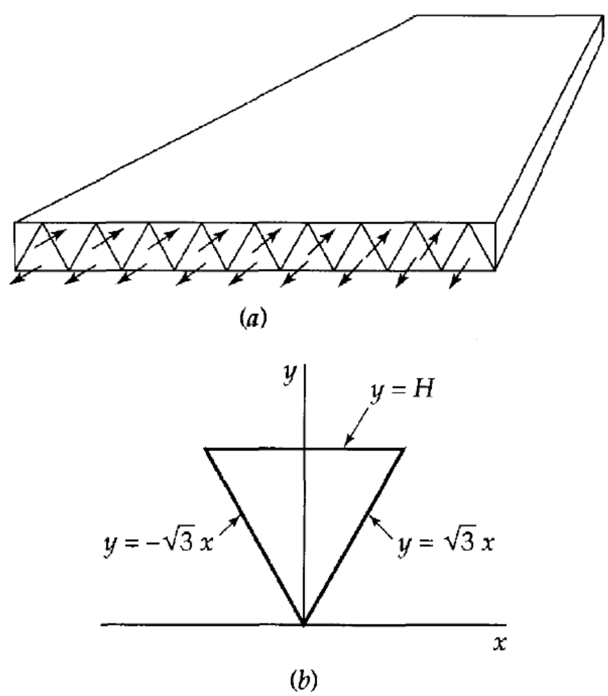

Fig. 3B.2. (*a*) Compact heat-exchanger element, showing channels of a triangular cross section; (*b*) coordinate system for an equilateral-triangular duct.

Solution

[2] An alternative formulation of the velocity profile is given by L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, Pergamon, Oxford, 2nd edition (1987), p. 54.

Part (a)

We assume that the fluid flows only in the z-direction and that the velocity varies as a function of x and y.

$$\mathbf{v} = v_z(x, y)\hat{\boldsymbol{z}}$$

If we assume the fluid does not slip on the walls, then it has the wall's velocity at $y = H$, $y = \sqrt{3}x$, and $y = -\sqrt{3}x$.

$$\begin{aligned} &\text{Boundary Condition 1:} && v_z(x, H) = 0 \\ &\text{Boundary Condition 2:} && v_z(x, \sqrt{3}x) = 0 \\ &\text{Boundary Condition 3:} && v_z(x, -\sqrt{3}x) = 0 \end{aligned}$$

The equation of continuity results by considering a mass balance over a volume element that the fluid is flowing through. Assuming the fluid density ρ is constant, the equation simplifies to

$$\nabla \cdot \mathbf{v} = 0. \tag{1}$$

The equation of motion results by considering a momentum balance over a volume element that the fluid is flowing through. Assuming the fluid viscosity μ is also constant, the equation simplifies to the Navier-Stokes equation.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla \cdot \rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g} \tag{2}$$

As this is a vector equation, it actually represents three scalar equations—one for each variable in the chosen coordinate system. Using Cartesian coordinates is the appropriate choice for this problem, so equations (1) and (2) will be used in (x, y, z). From Appendix B.4 on page 846, the continuity equation becomes

$$\underbrace{\frac{\partial v_x}{\partial x}}_{=\,0} + \underbrace{\frac{\partial v_y}{\partial y}}_{=\,0} + \underbrace{\frac{\partial v_z}{\partial z}}_{=\,0} = 0,$$

which doesn't tell us anything. From Appendix B.6 on page 848, the Navier-Stokes equation yields the following three scalar equations in Cartesian coordinates.

$$\begin{aligned}
\rho\Bigg(\underbrace{\frac{\partial v_x}{\partial t}}_{=\,0} + \underbrace{v_x\frac{\partial v_x}{\partial x}}_{=\,0} + \underbrace{v_y\frac{\partial v_x}{\partial y}}_{=\,0} + \underbrace{v_z\frac{\partial v_x}{\partial z}}_{=\,0}\Bigg) &= -\frac{\partial p}{\partial x} + \mu\Bigg[\underbrace{\frac{\partial^2 v_x}{\partial x^2}}_{=\,0} + \underbrace{\frac{\partial^2 v_x}{\partial y^2}}_{=\,0} + \underbrace{\frac{\partial^2 v_x}{\partial z^2}}_{=\,0}\Bigg] + \rho g_x \\
\rho\Bigg(\underbrace{\frac{\partial v_y}{\partial t}}_{=\,0} + \underbrace{v_x\frac{\partial v_y}{\partial x}}_{=\,0} + \underbrace{v_y\frac{\partial v_y}{\partial y}}_{=\,0} + \underbrace{v_z\frac{\partial v_y}{\partial z}}_{=\,0}\Bigg) &= -\frac{\partial p}{\partial y} + \mu\Bigg[\underbrace{\frac{\partial^2 v_y}{\partial x^2}}_{=\,0} + \underbrace{\frac{\partial^2 v_y}{\partial y^2}}_{=\,0} + \underbrace{\frac{\partial^2 v_y}{\partial z^2}}_{=\,0}\Bigg] + \rho g_y \\
\rho\Bigg(\underbrace{\frac{\partial v_z}{\partial t}}_{=\,0} + \underbrace{v_x\frac{\partial v_z}{\partial x}}_{=\,0} + \underbrace{v_y\frac{\partial v_z}{\partial y}}_{=\,0} + \underbrace{v_z\frac{\partial v_z}{\partial z}}_{=\,0}\Bigg) &= -\frac{\partial p}{\partial z} + \mu\Bigg[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \underbrace{\frac{\partial^2 v_z}{\partial z^2}}_{=\,0}\Bigg] + \rho g_z
\end{aligned}$$

The relevant equation for the velocity is the z-equation, which has simplified considerably from the assumption that $\mathbf{v} = v_z(x, y)\hat{\boldsymbol{z}}$.

$$0 = -\frac{\partial p}{\partial z} + \mu\left[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2}\right] + \rho g_z$$

The sum of $-\partial p/\partial z$ and ρg_z is the modified pressure gradient across the duct.

$$0 = -\frac{(\mathscr{P}_L - \mathscr{P}_0)}{L - 0} + \mu\left[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2}\right]$$

The velocity distribution thus satisfies the following PDE.

$$\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} = \frac{(\mathscr{P}_L - \mathscr{P}_0)}{\mu L}$$

We're asked to verify that the solution for it and its associated boundary conditions is

$$\begin{aligned} v_z &= \frac{(\mathscr{P}_0 - \mathscr{P}_L)}{4\mu L H}(y - H)(3x^2 - y^2) \\ &= \frac{(\mathscr{P}_0 - \mathscr{P}_L)}{4\mu L H}(3x^2 y - y^3 - 3Hx^2 + Hy^2). \end{aligned}$$

Find the second derivatives of v_z with respect to x and y.

$$\begin{aligned} \frac{\partial v_z}{\partial x} &= \frac{(\mathscr{P}_0 - \mathscr{P}_L)}{4\mu L H}(6xy - 6Hx) \\ \frac{\partial^2 v_z}{\partial x^2} &= \frac{(\mathscr{P}_0 - \mathscr{P}_L)}{4\mu L H}(6y - 6H) \\ \frac{\partial v_z}{\partial y} &= \frac{(\mathscr{P}_0 - \mathscr{P}_L)}{4\mu L H}(3x^2 - 3y^2 + 2Hy) \\ \frac{\partial^2 v_z}{\partial y^2} &= \frac{(\mathscr{P}_0 - \mathscr{P}_L)}{4\mu L H}(-6y + 2H) \end{aligned}$$

Adding them together, we have

$$\begin{aligned} \frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} &= \frac{(\mathscr{P}_0 - \mathscr{P}_L)}{4\mu L H}(6y - 6H - 6y + 2H) \\ &= \frac{(\mathscr{P}_0 - \mathscr{P}_L)}{4\mu L H}(-4H) \\ &= \frac{(\mathscr{P}_L - \mathscr{P}_0)}{\mu L}, \end{aligned}$$

so the velocity distribution satisfies the PDE. Setting $y = H$ or $y = \pm\sqrt{3}x$ gives $v_z = 0$, which means the boundary conditions are satisfied as well.

Part (b)

Average Velocity

To find the average velocity, integrate the velocity distribution over the cross-sectional area that the fluid flows through and then divide by that area.

$$\langle v_z \rangle = \frac{\int v_z \, dA}{\int dA}$$

For the equilateral triangle in Fig. 3B.2(b), $dA = 2x\,dy = \frac{2y}{\sqrt{3}}\,dy$.

$$\langle v_z \rangle = \frac{\iint v_z(x,y)\,dx\,dy}{\int_0^H \frac{2y}{\sqrt{3}}\,dy} = \frac{\iint v_z(x,y)\,dx\,dy}{\frac{2}{\sqrt{3}}\frac{H^2}{2}} = \frac{\sqrt{3}}{H^2}\iint v_z(x,y)\,dx\,dy$$

$$= \frac{\sqrt{3}}{H^2}\iint \frac{(\mathscr{P}_0 - \mathscr{P}_L)}{4\mu LH}(y-H)(3x^2 - y^2)\,dx\,dy$$

$$= \frac{\sqrt{3}(\mathscr{P}_0 - \mathscr{P}_L)}{4\mu LH^3}\iint (y-H)(3x^2-y^2)\,dx\,dy$$

Now the double integral will be evaluated over the triangle.

$$\langle v_z \rangle = \frac{\sqrt{3}(\mathscr{P}_0 - \mathscr{P}_L)}{4\mu LH^3}\left[\int_0^H \int_{-y/\sqrt{3}}^0 (y-H)(3x^2-y^2)\,dx\,dy + \int_0^H \int_0^{y/\sqrt{3}} (y-H)(3x^2-y^2)\,dx\,dy\right]$$

$$= \frac{\sqrt{3}(\mathscr{P}_0 - \mathscr{P}_L)}{4\mu LH^3}\left[\int_0^H (y-H)(x^3 - xy^2)\Big|_{-y/\sqrt{3}}^0\,dy + \int_0^H (y-H)(x^3-xy^2)\Big|_0^{y/\sqrt{3}}\,dy\right]$$

$$= \frac{\sqrt{3}(\mathscr{P}_0 - \mathscr{P}_L)}{4\mu LH^3}\left[\int_0^H (y-H)\left(\frac{y^3}{\sqrt{27}} - \frac{y^3}{\sqrt{3}}\right)dy + \int_0^H (y-H)\left(\frac{y^3}{\sqrt{27}} - \frac{y^3}{\sqrt{3}}\right)dy\right]$$

$$= \frac{\sqrt{3}(\mathscr{P}_0 - \mathscr{P}_L)}{4\mu LH^3}\left[2\int_0^H (y-H)\left(\frac{y^3}{\sqrt{27}} - \frac{y^3}{\sqrt{3}}\right)dy\right]$$

$$= \frac{\sqrt{3}(\mathscr{P}_0 - \mathscr{P}_L)}{4\mu LH^3}\left[-\frac{4}{3\sqrt{3}}\int_0^H (y^4 - Hy^3)\,dy\right]$$

$$= \frac{\sqrt{3}(\mathscr{P}_0 - \mathscr{P}_L)}{4\mu LH^3}\left[-\frac{4}{3\sqrt{3}}\cdot\left(-\frac{H^5}{20}\right)\right]$$

Therefore, the average velocity in the triangular duct is

$$\boxed{\langle v_z \rangle = \frac{(\mathscr{P}_0 - \mathscr{P}_L)H^2}{60\mu L}.}$$

Maximum Velocity

To find the absolute maximum of $v_z(x,y)$ over the triangular duct, find the function's critical points, that is, where the first derivatives with respect to x and y vanish.

$$\frac{\partial v_z}{\partial x} = \frac{(\mathscr{P}_0 - \mathscr{P}_L)}{4\mu LH}(6xy - 6Hx) = 0$$

$$\frac{\partial v_z}{\partial y} = \frac{(\mathscr{P}_0 - \mathscr{P}_L)}{4\mu LH}(3x^2 - 3y^2 + 2Hy) = 0$$

The first equation is satisfied if $x = 0$ or $y = H$. Plugging $x = 0$ into the second equation gives

$$\frac{(\mathscr{P}_0 - \mathscr{P}_L)}{4\mu LH}(-3y^2 + 2Hy) = 0,$$

which is satisfied if and only if $-3y^2 + 2Hy = 0$ or $y = \{0, 2H/3\}$. Two critical points are consequently $(0,0)$ and $(0, 2H/3)$. Plugging $y = H$ into the second equation gives

$$\frac{(\mathscr{P}_0 - \mathscr{P}_L)}{4\mu LH}(3x^2 - H^2) = 0,$$

which is satisfied if and only if $3x^2 - H^2 = 0$ or $x = \{-H/\sqrt{3}, H/\sqrt{3}\}$. Two more critical points are then $(-H/\sqrt{3}, H)$ and $(H/\sqrt{3}, H)$. $(0,0)$, $(-H/\sqrt{3}, H)$, and $(H/\sqrt{3}, H)$ all lie on the boundary of the triangular duct, where the velocity is assumed to be zero. As a result, the maximum must be at $(0, 2H/3)$.

$$\begin{aligned} v_{z,\max} = v_z(0, 2H/3) &= \frac{(\mathscr{P}_0 - \mathscr{P}_L)}{4\mu LH}\left(\frac{2H}{3} - H\right)\left(-\frac{4H^2}{9}\right) \\ &= \frac{(\mathscr{P}_0 - \mathscr{P}_L)}{4\mu LH}\left(-\frac{H}{3}\right)\left(-\frac{4H^2}{9}\right) \end{aligned}$$

Therefore, the maximum velocity in the triangular duct is

$$\boxed{v_{z,\max} = \frac{(\mathscr{P}_0 - \mathscr{P}_L)H^2}{27\mu L}.}$$

<u>**Mass Flow Rate**</u>

The volumetric flow rate is given by the integral of the velocity field over the cross-sectional area the fluid is flowing through.

$$\frac{dV}{dt} = \iint v_z \, dA$$

To get the mass flow rate, multiply both sides by the fluid density ρ.

$$\rho \frac{dV}{dt} = \rho \iint v_z(x, y)\, dx\, dy$$

Since ρ is assumed to be constant, it can be brought inside the derivative on the left side. Density times volume gives mass.

$$\begin{aligned} \frac{dm}{dt} &= \rho \iint v_z(x, y)\, dx\, dy \\ &= \rho \iint \frac{(\mathscr{P}_0 - \mathscr{P}_L)}{4\mu LH}(y - H)(3x^2 - y^2)\, dx\, dy \\ &= \frac{(\mathscr{P}_0 - \mathscr{P}_L)\rho}{4\mu LH} \iint (y - H)(3x^2 - y^2)\, dx\, dy \end{aligned}$$

This double integral was already evaluated when calculating the average velocity.

$$\begin{aligned} &= \frac{(\mathscr{P}_0 - \mathscr{P}_L)\rho}{4\mu LH}\left[-\frac{4}{3\sqrt{3}} \cdot \left(-\frac{H^5}{20}\right)\right] \\ &= \frac{(\mathscr{P}_0 - \mathscr{P}_L)H^4\rho}{60\sqrt{3}\mu L} \end{aligned}$$

Therefore, letting $w = dm/dt$, the mass flow rate in the triangular duct is

$$\boxed{w = \frac{\sqrt{3}(\mathscr{P}_0 - \mathscr{P}_L)H^4\rho}{180\mu L}.}$$