Problem 3B.4

Creeping flow between two concentric spheres (Fig. 3B.4). A very viscous Newtonian fluid flows in the space between two concentric spheres, as shown in the figure. It is desired to find the rate of flow in the system as a function of the imposed pressure difference. Neglect end effects and postulate that v_θ depends only on r and θ with the other velocity components zero.

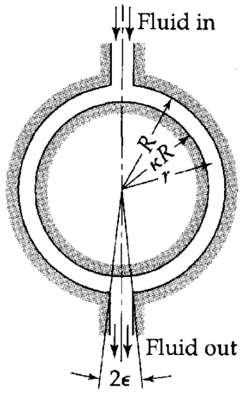

Fig. 3B.4. Creeping flow in the region between two stationary concentric spheres.

(a) Using the equation of continuity, show that $v_\theta \sin\theta = u(r)$, where $u(r)$ is a function of r to be determined.

(b) Write the θ-component of the equation of motion for this system, assuming the flow to be slow enough that the $[\mathbf{v} \cdot \nabla \mathbf{v}]$ term is negligible. Show that this gives

$$0 = -\frac{1}{r}\frac{\partial \mathscr{P}}{\partial \theta} + \mu \left[\frac{1}{\sin\theta}\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{du}{dr}\right)\right] \tag{3B.4-1}$$

(c) Separate this into two equations

$$\sin\theta \frac{\partial \mathscr{P}}{\partial \theta} = B; \qquad \frac{\mu}{r}\frac{d}{dr}\left(r^2 \frac{du}{dr}\right) = B \tag{3B.4-2, 3}$$

where B is the separation constant, and solve the two equations to get

$$B = \frac{\mathscr{P}_2 - \mathscr{P}_1}{2 \ln \cot \frac{1}{2}\varepsilon} \tag{3B.4-4}$$

$$u(r) = \frac{(\mathscr{P}_1 - \mathscr{P}_2)R}{4\mu \ln \cot(\varepsilon/2)} \left[\left(1 - \frac{r}{R}\right) + \kappa \left(1 - \frac{R}{r}\right)\right] \tag{3B.4-5}$$

where $\mathscr{P}_1$ and $\mathscr{P}_2$ are the values of the modified pressure at $\theta = \varepsilon$ and $\theta = \pi - \varepsilon$, respectively.

(d) Use the results above to get the mass rate of flow

$$w = \frac{\pi(\mathscr{P}_1 - \mathscr{P}_2)R^3(1-\kappa)^3\rho}{12\mu \ln \cot(\varepsilon/2)} \tag{3B.4-6}$$

Solution

Part (a)

For two concentric spheres a spherical coordinate system (r, θ, ϕ) is used, where θ represents the angle from the polar axis. We assume that the fluid flows only in the θ-direction and that the velocity varies as a function of r and θ.

$$\mathbf{v} = v_\theta(r, \theta)\hat{\boldsymbol{\theta}}$$

If we assume the fluid does not slip on the walls, then it has the wall's velocity at $r = \kappa R$ and $r = R$.

$$\begin{aligned} &\text{Boundary Condition 1:} \qquad && v_\theta(\kappa R, \theta) = 0 \\ &\text{Boundary Condition 2:} && v_\theta(R, \theta) = 0 \end{aligned}$$

The equation of continuity results by considering a mass balance over a volume element that the fluid is flowing through. Assuming the fluid density ρ is constant, the equation simplifies to

$$\nabla \cdot \mathbf{v} = 0.$$

From Appendix B.4 on page 846, the continuity equation in spherical coordinates becomes

$$\underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r)}_{=\,0} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta \sin\theta) + \underbrace{\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}}_{=\,0} = 0.$$

Multiply both sides by $r\sin\theta$.

$$\frac{\partial}{\partial\theta}(v_\theta \sin\theta) = 0$$

Integrate both sides partially with respect to θ, to obtain

$$\boxed{v_\theta \sin\theta = u(r),}$$

where $u(r)$ is an arbitrary function of r. Divide both sides by $\sin\theta$ to solve for v_θ.

$$v_\theta(r, \theta) = \frac{u(r)}{\sin\theta}$$

Use the boundary conditions for v_θ to obtain those for u.

$$\begin{aligned} v_\theta(\kappa R, \theta) &= \frac{u(\kappa R)}{\sin\theta} = 0 \qquad &&\to \qquad && u(\kappa R) = 0 \\ v_\theta(R, \theta) &= \frac{u(R)}{\sin\theta} = 0 &&\to && u(R) = 0 \end{aligned}$$

Part (b)

The equation of motion results by considering a momentum balance over a volume element that the fluid is flowing through. Assuming the fluid viscosity μ is constant in addition to ρ, the equation simplifies to the Navier-Stokes equation.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla \cdot \rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}$$

Creeping flow is assumed, so the acceleration terms on the left side are zero.

$$\mathbf{0} = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g}$$

As this is a vector equation, it actually represents three scalar equations—one for each variable in the chosen coordinate system. From Appendix B.6 on page 848, the Navier-Stokes equation yields the following three scalar equations in spherical coordinates.

$$0 = -\frac{\partial p}{\partial r} + \mu \left[\underbrace{\frac{1}{r^2} \frac{\partial^2}{\partial r^2}(r^2 v_r)}_{=0} + \underbrace{\frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial v_r}{\partial \theta}\right)}_{=0} + \underbrace{\frac{1}{r^2 \sin^2\theta} \frac{\partial^2 v_r}{\partial \phi^2}}_{=0} \right] + \rho g_r$$

$$0 = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(v_\theta \sin\theta)\right) + \underbrace{\frac{1}{r^2 \sin^2\theta}\frac{\partial^2 v_\theta}{\partial \phi^2}}_{=0} + \underbrace{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}}_{=0} - \underbrace{\frac{2\cot\theta}{r^2 \sin\theta}\frac{\partial v_\phi}{\partial \phi}}_{=0} \right] + \rho g_\theta$$

$$0 = \underbrace{-\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}}_{=0} + \mu \left[\underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial v_\phi}{\partial r}\right)}_{=0} + \underbrace{\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(v_\phi \sin\theta)\right)}_{=0} + \underbrace{\frac{1}{r^2 \sin^2\theta}\frac{\partial^2 v_\phi}{\partial \phi^2}}_{=0} + \underbrace{\frac{2}{r^2 \sin\theta}\frac{\partial v_r}{\partial \phi}}_{=0} + \underbrace{\frac{2\cot\theta}{r^2 \sin\theta}\frac{\partial v_\theta}{\partial \phi}}_{=0} \right] + \underbrace{\rho g_\phi}_{=0}$$

The relevant equation for the velocity is the θ-equation, which has simplified considerably from the assumption that $\mathbf{v} = v_\theta(r, \theta)\hat{\boldsymbol{\theta}}$.

$$0 = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(v_\theta \sin\theta)\right) \right] + \rho g_\theta$$

It was found from part (a) that $v_\theta \sin\theta$ is only a function of r, so the second term in square brackets is zero.

$$0 = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial v_\theta}{\partial r}\right) \right] + \rho g_\theta$$

Replace v_θ with $u(r)/\sin\theta$. Also, assuming that gravity points downward, the vector is $\mathbf{g} = -g\hat{\mathbf{z}} = -g[(\cos\theta)\hat{\mathbf{r}} + (-\sin\theta)\hat{\boldsymbol{\theta}}]$ in spherical coordinates, so $g_\theta = g\sin\theta$.

$$0 = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \rho g \sin\theta + \mu \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\frac{u(r)}{\sin\theta}\right) \right]$$

$$0 = -\frac{1}{r}\left(\frac{\partial p}{\partial \theta} - \rho g r \sin\theta\right) + \mu \left[\frac{1}{\sin\theta}\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{du}{dr}\right) \right]$$

Therefore,

$$\boxed{0 = -\frac{1}{r}\frac{\partial \mathscr{P}}{\partial \theta} + \mu \left[\frac{1}{\sin\theta}\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{du}{dr}\right) \right],}$$

where $\mathscr{P} = \mathscr{P}(r, \theta) = p(r, \theta) + \rho g r \cos\theta$ is the modified pressure.

Part (c)

Bring the term with $\mathscr{P}$ to the left side.

$$\frac{1}{r}\frac{\partial \mathscr{P}}{\partial \theta} = \frac{\mu}{r^2 \sin\theta}\frac{d}{dr}\left(r^2\frac{du}{dr}\right)$$

Multiply both sides by $r\sin\theta$.

$$\sin\theta\frac{\partial \mathscr{P}}{\partial \theta} = \frac{\mu}{r}\frac{d}{dr}\left(r^2\frac{du}{dr}\right)$$

If we assume that the space between the spheres is small, then the modified pressure is approximately constant in r.

$$\sin\theta\frac{d \mathscr{P}}{d \theta} = \frac{\mu}{r}\frac{d}{dr}\left(r^2\frac{du}{dr}\right)$$

The only way a function of θ can be equal to a function of r is if both are equal to a constant B.

$$\sin\theta\frac{d \mathscr{P}}{d \theta} = \frac{\mu}{r}\frac{d}{dr}\left(r^2\frac{du}{dr}\right) = B$$

Solve the first equation for $\mathscr{P}$.

$$\sin\theta\frac{d \mathscr{P}}{d \theta} = B$$

Multiply both sides by $d\theta/\sin\theta$.

$$d\mathscr{P} = \frac{B}{\sin\theta}\,d\theta$$

Integrate both sides.

$$\int_{\mathscr{P}_1}^{\mathscr{P}_2} d\mathscr{P} = \int_{\theta_1}^{\theta_2}\frac{B}{\sin\theta}\,d\theta,$$

where $\theta_1 = \varepsilon$, $\theta_2 = \pi - \varepsilon$, and $\mathscr{P}_1$ and $\mathscr{P}_2$ are the modified pressures at these angles, respectively.

$$\begin{aligned}
\mathscr{P}_2 - \mathscr{P}_1 &= B\ln\tan\frac{\theta}{2}\bigg|_{\varepsilon}^{\pi-\varepsilon} \\
&= B\left(\ln\tan\frac{\pi-\varepsilon}{2} - \ln\tan\frac{\varepsilon}{2}\right) \\
&= B\ln\frac{\tan\frac{\pi-\varepsilon}{2}}{\tan\frac{\varepsilon}{2}} \\
&= B\ln\frac{\cot\frac{\varepsilon}{2}}{\tan\frac{\varepsilon}{2}} \\
&= B\ln\left(\cot\frac{\varepsilon}{2}\right)^2 \\
&= 2B\ln\cot\frac{\varepsilon}{2}
\end{aligned}$$

Dividing both sides by $2\ln\cot(\varepsilon/2)$, therefore,

$$\boxed{B = \frac{\mathscr{P}_2 - \mathscr{P}_1}{2\ln\cot(\varepsilon/2)}.}$$

Now solve the second equation for u.

$$\frac{\mu}{r}\frac{d}{dr}\left(r^2\frac{du}{dr}\right) = B$$

Multiply both sides by r/μ.

$$\frac{d}{dr}\left(r^2\frac{du}{dr}\right) = \frac{Br}{\mu}$$

Integrate both sides with respect to r.

$$r^2\frac{du}{dr} = \frac{Br^2}{2\mu} + C_1$$

Divide both sides by r^2.

$$\frac{du}{dr} = \frac{B}{2\mu} + \frac{C_1}{r^2}$$

Integrate both sides with respect to r once more.

$$u(r) = \frac{B}{2\mu}r - \frac{C_1}{r} + C_2$$

Apply the boundary conditions here to determine C_1 and C_2.

$$\begin{aligned} u(\kappa R) &= \frac{B}{2\mu}\kappa R - \frac{C_1}{\kappa R} + C_2 = 0 \\ u(R) &= \frac{B}{2\mu}R - \frac{C_1}{R} + C_2 = 0 \end{aligned}$$

Solving this system of equations yields

$$C_1 = -\frac{B\kappa R^2}{2\mu} \quad \text{and} \quad C_2 = -BR\frac{\kappa+1}{2\mu}.$$

So then

$$\begin{aligned} u(r) &= \frac{B}{2\mu}r + \frac{B\kappa R^2}{2\mu}\frac{1}{r} - BR\frac{\kappa+1}{2\mu} \\ &= \frac{BR}{2\mu}\left(\frac{r}{R} + \frac{\kappa R}{r} - \kappa - 1\right) \\ &= -\frac{BR}{2\mu}\left[\left(1-\frac{r}{R}\right) + \kappa\left(1-\frac{R}{r}\right)\right] \\ &= -\frac{\mathscr{P}_2 - \mathscr{P}_1}{2\ln\cot(\varepsilon/2)}\frac{R}{2\mu}\left[\left(1-\frac{r}{R}\right) + \kappa\left(1-\frac{R}{r}\right)\right] \end{aligned}$$

Therefore,

$$\boxed{u(r) = \frac{(\mathscr{P}_1 - \mathscr{P}_2)R}{4\mu\ln\cot(\varepsilon/2)}\left[\left(1-\frac{r}{R}\right) + \kappa\left(1-\frac{R}{r}\right)\right].}$$

Part (d)

The volumetric flow rate is obtained by integrating the velocity distribution over the area the fluid is flowing through.

$$\frac{dV}{dt} = \iint v_\theta \, dA$$

To get the mass flow rate, multiply both sides by the density ρ.

$$\rho \frac{dV}{dt} = \rho \iint v_\theta \, dA$$

Bring ρ inside the derivative.

$$\frac{d(\rho V)}{dt} = \rho \iint v_\theta \, dA$$

Density times volume is mass.

$$\begin{aligned}
\frac{dm}{dt} &= \rho \iint v_\theta(r,\theta)\, dA \\
&= \rho \int_0^{2\pi} \int_{\kappa R}^{R} v_\theta(r,\theta)(dr)(r \sin\theta \, d\phi) \\
&= \rho \int_0^{2\pi} \int_{\kappa R}^{R} \frac{u(r)}{\sin\theta}(r \sin\theta \, dr \, d\phi) \\
&= \rho \int_0^{2\pi} \int_{\kappa R}^{R} r u(r) \, dr \, d\phi \\
&= \rho \left(\int_0^{2\pi} d\phi \right) \int_{\kappa R}^{R} r u(r)\, dr \\
&= 2\pi\rho \int_{\kappa R}^{R} r u(r)\, dr \\
&= 2\pi\rho \int_{\kappa R}^{R} r \frac{(\mathscr{P}_1 - \mathscr{P}_2)R}{4\mu \ln \cot(\varepsilon/2)} \left[\left(1 - \frac{r}{R}\right) + \kappa \left(1 - \frac{R}{r}\right) \right] dr \\
&= \frac{\pi(\mathscr{P}_1 - \mathscr{P}_2)R\rho}{2\mu \ln\cot(\varepsilon/2)} \int_{\kappa R}^{R} \left(r - \frac{r^2}{R} + \kappa r - \kappa R \right) dr \\
&= \frac{\pi(\mathscr{P}_1 - \mathscr{P}_2)R\rho}{2\mu \ln\cot(\varepsilon/2)} \cdot \frac{R^2}{6}(1-\kappa)^3
\end{aligned}$$

Therefore, letting $w = dm/dt$, the mass flow rate is

$$\boxed{w = \frac{\pi(\mathscr{P}_1 - \mathscr{P}_2)R^3(1-\kappa)^3\rho}{12\mu \ln\cot(\varepsilon/2)}.}$$