Problem 3B.5

Parallel-disk viscometer (Fig. 3B.5). A fluid, whose viscosity is to be measured, is placed in the gap of thickness B between the two disks of radius R. One measures the torque T_z required to turn the upper disk at an angular velocity Ω. Develop the formula for deducing the viscosity from these measurements. Assume creeping flow.

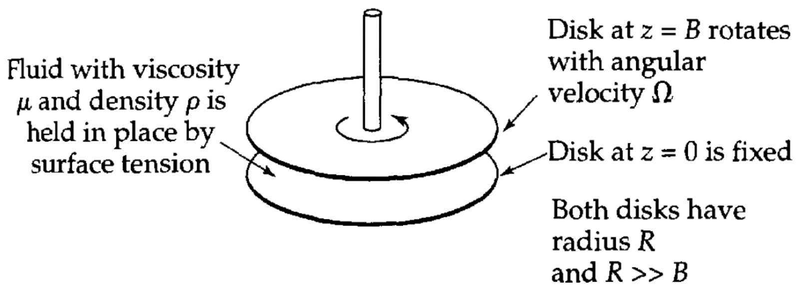

Fig. 3B.5. Parallel-disk viscometer.

(a) Postulate that for small values of Ω the velocity profiles have the form $v_r = 0$, $v_z = 0$, and $v_\theta = rf(z)$; why does this form for the tangential velocity seem reasonable? Postulate further that $\mathscr{P} = \mathscr{P}(r, z)$. Write down the resulting simplified equations of continuity and motion.

(b) From the θ-component of the equation of motion, obtain a differential equation for $f(z)$. Solve the equation for $f(z)$ and evaluate the constants of integration. This leads ultimately to the result $v_\theta = \Omega r(z/B)$. Could you have guessed this result?

(c) Show that the desired working equation for deducing the viscosity is $\mu = 2BT_z/\pi\Omega R^4$.

(d) Discuss the advantages and disadvantages of this instrument.

Solution

Part (a)

A cylindrical coordinate system will be used for this problem. We assume that the fluid flows only in the θ-direction and that the velocity varies as a function of r and z.

$$\mathbf{v} = v_\theta(r, z)\hat{\boldsymbol{\theta}}$$

If Ω is small, then the fluid will not slip on the disk at $z = B$, meaning that the fluid will have the disk's tangential velocity at $z = B$. Assuming that no slipping occurs on the disk at $z = 0$ either, the velocity here will be zero because this disk is stationary. These ideas lead to the following boundary conditions.

Boundary Condition 1:	$v_\theta(r, 0) = 0$
Boundary Condition 2:	$v_\theta(r, B) = \Omega r$

The equation of continuity results by considering a mass balance over a volume element that the fluid is flowing through. Assuming the fluid density ρ is constant, the equation simplifies to

$$\nabla \cdot \mathbf{v} = 0. \tag{1}$$

The equation of motion results by considering a momentum balance over a volume element that the fluid is flowing through. Assuming the fluid viscosity μ is also constant, the equation simplifies to the Navier-Stokes equation.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla\cdot\rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g} \tag{2}$$

Creeping flow is assumed, so the acceleration terms on the left side are zero.

$$\mathbf{0} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g} \tag{2}$$

As this is a vector equation, it actually represents three scalar equations—one for each variable in the chosen coordinate system. From Appendix B.4 on page 846, the continuity equation becomes

$$\underbrace{\frac{1}{r}\frac{\partial}{\partial r}(rv_r)}_{=0} + \underbrace{\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}}_{=0} + \underbrace{\frac{\partial v_z}{\partial z}}_{=0} = 0,$$

which doesn't tell us anything. From Appendix B.6 on page 848, the Navier-Stokes equation yields the following three scalar equations in cylindrical coordinates.

$$0 = -\frac{\partial p}{\partial r} + \mu\left[\underbrace{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)}_{=0} + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_r}{\partial z^2}}_{=0} - \underbrace{\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}}_{=0}\right] + \underbrace{\rho g_r}_{=0}$$

$$0 = \underbrace{-\frac{1}{r}\frac{\partial p}{\partial\theta}}_{=0} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}}_{=0} + \frac{\partial^2 v_\theta}{\partial z^2} + \underbrace{\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}}_{=0}\right] + \underbrace{\rho g_\theta}_{=0}$$

$$0 = -\frac{\partial p}{\partial z} + \mu\left[\underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)}_{=0} + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial\theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_z}{\partial z^2}}_{=0}\right] + \rho g_z$$

The relevant equation for the velocity is the θ-equation, which has simplified considerably from the assumption that $\mathbf{v} = v_\theta(r,z)\hat{\boldsymbol{\theta}}$.

$$0 = \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \frac{\partial^2 v_\theta}{\partial z^2}\right]$$

Part (b)

Because $R \gg B$, the gap between the disks is very small. The fluid velocity can then be assumed to be similar to that of the disk, varying with distance from the axis of rotation.

$$v_\theta(r,z) = rf(z)$$

This form will be substituted into the θ-equation to determine $f(z)$—how the angular velocity varies with height.

$$0 = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r^2 f)\right) + \frac{\partial^2}{\partial z^2}(rf)$$

$$0 = f\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(r^2)\right) + r\frac{d^2 f}{dz^2}$$

The first term evaluates to zero.

$$0 = r\frac{d^2 f}{dz^2}$$

Divide both sides by r.

$$\frac{d^2 f}{dz^2} = 0$$

Integrate both sides with respect to z.

$$\frac{df}{dz} = C_1$$

Integrate both sides with respect to z once more.

$$f(z) = C_1 z + C_2$$

Substitute the form for $v_\theta(r, z)$ into the boundary conditions to find those for f.

$$\begin{aligned} v_\theta(r,0) &= 0 & &\rightarrow & rf(0) &= 0 & &\rightarrow & f(0) &= 0 \\ v_\theta(r,B) &= \Omega r & &\rightarrow & rf(B) &= \Omega r & &\rightarrow & f(B) &= \Omega \end{aligned}$$

Now C_1 and C_2 can be determined.

$$\begin{aligned} f(0) &= C_2 = 0 \\ f(B) &= C_1 B + C_2 = \Omega \end{aligned}$$

Since $C_2 = 0$, the second equation reduces to $C_1 B = \Omega$, so $C_1 = \Omega/B$. So then

$$f(z) = \frac{\Omega}{B} z.$$

Therefore,

$$v_\theta(r, z) = \Omega r \frac{z}{B}.$$

I don't think I could have guessed this, but that's okay because it can be derived.

Part (c)

$\tau_{z\theta}$ represents the viscous force per unit area in the θ-direction on a plane perpendicular to the z-direction. It's given in Table B.1 on page 844.

$$\begin{aligned} \tau_{z\theta} &= -\mu \frac{\partial v_\theta}{\partial z} \\ &= -\frac{\mu\Omega}{B} r \end{aligned}$$

To obtain the total viscous force of the fluid acting on the disk, $\tau_{z\theta}$ needs to be integrated over the surface area of the disk. Multiplying the integrand by the distance from the axis of rotation (the moment arm) gives the torque.

$$T = \int \tau_{z\theta}\big|_{z=B} \cdot r\, dA$$

The sign of $\tau_{z\theta}$ is positive because the fluid is acting on a surface of greater z.

$$\begin{aligned} T &= \int_0^R \left(-\frac{\mu\Omega}{B}r\right) \cdot r(2\pi r\,dr) \\ &= -\frac{2\pi\mu\Omega}{B}\int_0^R r^3\,dr \\ &= -\frac{2\pi\mu\Omega}{B} \cdot \frac{R^4}{4} \\ &= -\frac{\pi\mu\Omega R^4}{2B} \end{aligned}$$

Consequently, the torque that needs to be applied to maintain the angular velocity Ω is

$$T_z = \frac{\pi\mu\Omega R^4}{2B}.$$

Therefore,

$$\mu = \frac{2BT_z}{\pi\Omega R^4}.$$

Part (d)

The advantage of this instrument is that it's fairly simple to use and calculate μ. The disadvantage is that the accuracy of this number depends on whether all the assumptions made are legitimate.