Problem 3B.9

Slow transverse flow around a cylinder (see Fig. 3.7-1). An incompressible Newtonian fluid approaches a stationary cylinder with a uniform, steady velocity v_∞ in the positive x direction. When the equations of change are solved for creeping flow, the following expressions[5] are found for the pressure and velocity in the immediate vicinity of the cylinder (they are *not* valid at large distances):

$$p(r,\theta) = p_\infty - C\mu \frac{v_\infty \cos\theta}{r} - \rho g r \sin\theta \tag{3B.9-1}$$

$$v_r = C v_\infty \left[\frac{1}{2}\ln\left(\frac{r}{R}\right) - \frac{1}{4} + \frac{1}{4}\left(\frac{R}{r}\right)^2\right]\cos\theta \tag{3B.9-2}$$

$$v_\theta = -C v_\infty \left[\frac{1}{2}\ln\left(\frac{r}{R}\right) + \frac{1}{4} - \frac{1}{4}\left(\frac{R}{r}\right)^2\right]\sin\theta \tag{3B.9-3}$$

Here p_∞ is the pressure far from the cylinder at $y = 0$ and

$$C = \frac{2}{\ln(7.4/\text{Re})} \tag{3B.9-4}$$

with the Reynolds number defined as $\text{Re} = 2Rv_\infty\rho/\mu$.

(a) Use these results to get the pressure p, the shear stress $\tau_{r\theta}$, and the normal stress τ_{rr} at the surface of the cylinder.

(b) Show that the x-component of the force per unit area exerted by the liquid on the cylinder is

$$-p|_{r=R}\cos\theta + \tau_{r\theta}|_{r=R}\sin\theta \tag{3B.9-5}$$

(c) Obtain the force $F_x = 2C\pi L\mu v_\infty$ exerted in the x direction on a length L of the cylinder.

Solution

Fig. 3.7-1 is shown below, which illustrates the flow and the chosen coordinate system.

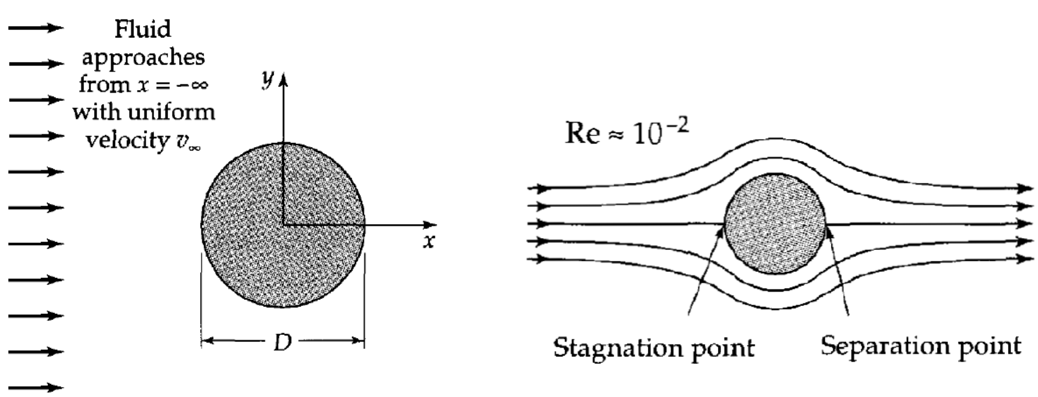


Fig. 3.7-1. Transverse flow around a cylinder.

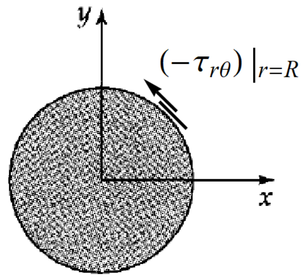
[5] See G. K. Batchelor, *An Introduction to Fluid Dynamics*, Cambridge University Press (1967), pp. 244–246, 261.

Part (a)

The surface of the cylinder is at $r = R$, so that's where the quantities of interest will be evaluated at. The pressure on the surface is

$$p(R,\theta) = p_\infty - C\mu\frac{v_\infty \cos\theta}{R} - \rho g R \sin\theta.$$

The shear stress $\tau_{r\theta}$ represents the viscous force in the θ-direction on a unit area perpendicular to the r-direction. Since the fluid here is in a region of greater r acting on a surface of lesser r, a minus sign is needed in front of $\tau_{r\theta}$. The shear stress on the cylinder's surface is then $-\tau_{r\theta}$ evaluated at $r = R$.

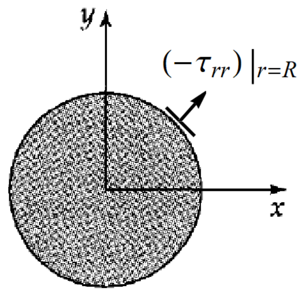


Use the formula for $\tau_{r\theta}$ in Appendix B.1 on page 844 and substitute the given functions for v_r and v_θ.

$$\begin{aligned}
(-\tau_{r\theta})|_{r=R} &= \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]\Bigg|_{r=R} \\
&= \mu\Bigg\{r(-Cv_\infty)\frac{\partial}{\partial r}\left[\frac{1}{2r}\ln\left(\frac{r}{R}\right) + \frac{1}{4r} - \frac{1}{4r}\left(\frac{R}{r}\right)^2\right]\sin\theta \\
&\qquad + \frac{1}{r}(Cv_\infty)\left[\frac{1}{2}\ln\left(\frac{r}{R}\right) - \frac{1}{4} + \frac{1}{4}\left(\frac{R}{r}\right)^2\right]\frac{\partial}{\partial\theta}(\cos\theta)\Bigg\}\Bigg|_{r=R} \\
&= \mu\Bigg\{r(-Cv_\infty)\left[-\frac{1}{2r^2}\ln\left(\frac{r}{R}\right) + \frac{1}{2r}\frac{1}{\frac{r}{R}}\cdot\frac{1}{R} - \frac{1}{4r^2} + \frac{3R^2}{4r^4}\right]\sin\theta \\
&\qquad + \frac{1}{r}(Cv_\infty)\left[\frac{1}{2}\ln\left(\frac{r}{R}\right) - \frac{1}{4} + \frac{1}{4}\left(\frac{R}{r}\right)^2\right](-\sin\theta)\Bigg\}\Bigg|_{r=R} \\
&= \mu\Bigg[R(-Cv_\infty)\left(0 + \frac{1}{2R^2} - \frac{1}{4R^2} + \frac{3}{4R^2}\right)\sin\theta \\
&\qquad + \frac{1}{R}(Cv_\infty)\left(0 - \frac{1}{4} + \frac{1}{4}\right)(-\sin\theta)\Bigg] \\
&= \mu R(-Cv_\infty)\left(\frac{1}{R^2}\right)\sin\theta \\
&= -\frac{C\mu v_\infty}{R}\sin\theta
\end{aligned}$$

Notice that for $0 < \theta < \pi$, the shear stress is negative because the fluid is flowing over the cylinder in the negative θ-direction; however, for $\pi < \theta < 2\pi$, the shear stress is positive because the fluid is flowing under the cylinder in the positive θ-direction.

The normal stress τ_{rr} represents the viscous force in the r-direction on a unit area perpendicular to the r-direction. As with $\tau_{r\theta}$, a minus sign is needed in front because the fluid is in a region of greater r acting on a surface of lesser r. The normal stress on the cylinder's surface due to viscosity is then $-\tau_{rr}$ evaluated at $r = R$.



Use the formula for τ_{rr} in Appendix B.1 on page 844.

$$(-\tau_{rr})|_{r=R} = \left.\left\{\mu\left[2\frac{\partial v_r}{\partial r}\right] + \left(\frac{2}{3}\mu - \kappa\right)(\nabla\cdot\mathbf{v})\right\}\right|_{r=R}$$

Since the fluid is incompressible, $\nabla\cdot\mathbf{v} = 0$. Substitute the given function for v_r and simplify the result.

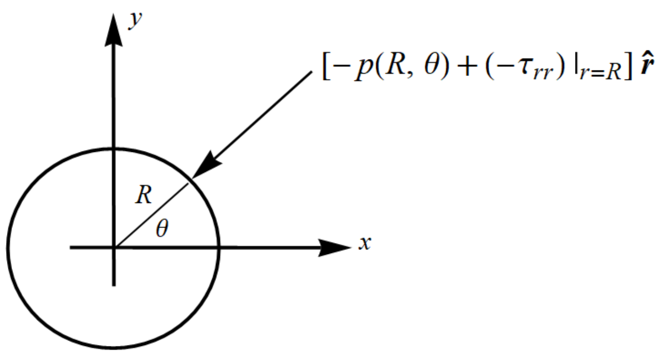
$$\begin{aligned}
(-\tau_{rr})|_{r=R} &= 2\mu\left.\frac{\partial v_r}{\partial r}\right|_{r=R} \\
&= 2\mu\frac{\partial}{\partial r}\left.\left\{Cv_\infty\left[\frac{1}{2}\ln\left(\frac{r}{R}\right) - \frac{1}{4} + \frac{1}{4}\left(\frac{R}{r}\right)^2\right]\cos\theta\right\}\right|_{r=R} \\
&= 2\mu Cv_\infty\left.\left(\frac{1}{2}\frac{1}{\frac{r}{R}}\cdot\frac{1}{R} - \frac{1}{2}\frac{R^2}{r^3}\right)\right|_{r=R}\cos\theta \\
&= 2\mu Cv_\infty\left(\frac{1}{2R} - \frac{1}{2R}\right)\cos\theta \\
&= 0
\end{aligned}$$

Part (b)

The total force acting on the cylinder is due to the pressure, the viscous shear stress, and the viscous normal stress. It can be split into two components, the first acting normally and the second acting tangentially.

$$\mathbf{F} = \mathbf{F}_\perp + \mathbf{F}_\parallel$$

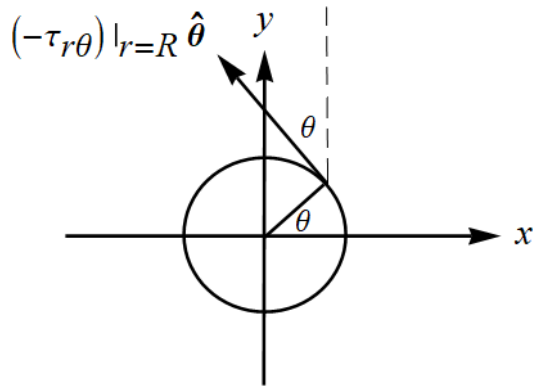
The force per unit area acting normally is due to the pressure and viscous normal stress as shown below.



Integrate it over the lateral surface area of the cylinder to get the force. An extra factor of $\cos\theta$ is needed in the integrand to get the x-component of force specifically. Note that p has a minus sign in front of it because the force resulting from it acts radially inward. No second minus sign is needed in front of τ_{rr} because the velocity components that it's in terms of already take care of it.

$$\begin{aligned}
(F_\perp)_x &= \int [-p(R,\theta) + (-\tau_{rr})|_{r=R}]\hat{\mathbf{r}}(\cos\theta)\cdot d\mathbf{A} \\
&= \int [-p(R,\theta)]\hat{\mathbf{r}}(\cos\theta)\cdot(\hat{\mathbf{r}}\,dA) \\
&= -\int p(R,\theta)(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}})\cos\theta\,dA \\
&= -\int p(R,\theta)\cos\theta\,dA \\
&= -\int \left[p_\infty - C\mu\frac{v_\infty\cos\theta}{R} - \rho g R\sin\theta\right]\cos\theta\,(L\,ds) \\
&= -\int_0^{2\pi} \left[p_\infty\cos\theta - C\mu\frac{v_\infty\cos^2\theta}{R} - \rho g R\sin\theta\cos\theta\right][L(R\,d\theta)] \\
&= LR\Bigg(-p_\infty \underbrace{\int_0^{2\pi}\cos\theta\,d\theta}_{=0} + C\mu\frac{v_\infty}{R}\underbrace{\int_0^{2\pi}\cos^2\theta\,d\theta}_{=\pi} + \rho g R\underbrace{\int_0^{2\pi}\sin\theta\cos\theta\,d\theta}_{=0}\Bigg) \\
&= C\mu v_\infty L\pi
\end{aligned}$$

The force per unit area acting tangentially is due only to the viscous shear stress as shown below.



Integrate it over the lateral surface area of the cylinder to get the force. An extra factor of $|\sin\theta|$ is needed in the integrand to get the x-component of force specifically. The absolute value sign is included because the flow is the same above and below the x-axis.

$$\begin{aligned}
(F_{\parallel})_x &= \int [(-\tau_{r\theta})|_{r=R}]\hat{\boldsymbol{\theta}}|\sin\theta| \cdot d\mathbf{A} \\
&= \int [(-\tau_{r\theta})|_{r=R}]\hat{\boldsymbol{\theta}}|\sin\theta| \cdot (L\,d\mathbf{s}) \\
&= \int_0^{2\pi} [(-\tau_{r\theta})|_{r=R}]\hat{\boldsymbol{\theta}}|\sin\theta| \cdot [L(R\,d\boldsymbol{\theta})] \\
&= LR\int_0^{2\pi} (-\tau_{r\theta})|_{r=R}\,|\sin\theta|\,(\hat{\boldsymbol{\theta}} \cdot d\boldsymbol{\theta})
\end{aligned}$$

The fluid flows over the top of the cylinder in the negative θ-direction, so the dot product yields a minus sign from 0 to π. From π to 2π, however, the fluid flows under the cylinder in the positive θ-direction, so the dot product yields no minus sign.

$$\begin{aligned}
(F_{\parallel})_x &= LR\left[\int_0^{\pi} (-\tau_{r\theta})|_{r=R}\,(\sin\theta)\,(-d\theta) + \int_{\pi}^{2\pi} (-\tau_{r\theta})|_{r=R}\,(-\sin\theta)\,(d\theta)\right] \\
&= LR\left[-\int_0^{\pi}\left(-\frac{C\mu v_\infty}{R}\sin\theta\right)\sin\theta\,d\theta - \int_{\pi}^{2\pi}\left(-\frac{C\mu v_\infty}{R}\sin\theta\right)\sin\theta\,d\theta\right] \\
&= LR\left[\frac{C\mu v_\infty}{R}\underbrace{\int_0^{\pi}\sin^2\theta\,d\theta}_{=\pi/2} + \frac{C\mu v_\infty}{R}\underbrace{\int_{\pi}^{2\pi}\sin^2\theta\,d\theta}_{=\pi/2}\right] \\
&= C\mu v_\infty L\pi
\end{aligned}$$

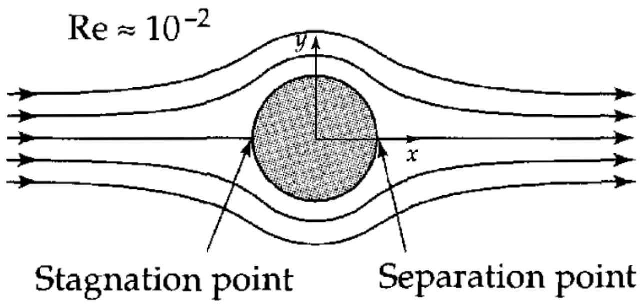
Part (c)

Therefore, the total force in the x-direction on the cylinder is

$$\begin{aligned}
F_x &= (F_{\perp})_x + (F_{\parallel})_x \\
&= C\mu v_\infty L\pi + C\mu v_\infty L\pi \\
&= 2C\mu v_\infty L\pi.
\end{aligned}$$

Part (d)

Here the formulas for the pressure and velocity for the flow around a cylinder will be derived.



The velocity is assumed to have radial and angular components that both vary with r and θ.

$$\mathbf{v} = v_r(r,\theta)\hat{\mathbf{r}} + v_\theta(r,\theta)\hat{\boldsymbol{\theta}}$$

In addition, the pressure is assumed to vary at each point in the plane.

$$p = p(r,\theta)$$

One boundary condition is obtained from the assumption that no fluid crosses the cylindrical surface (that is, it's impermeable), and a second is obtained from the assumption that the fluid does not slip on the cylindrical surface.

$$\begin{aligned} \text{Boundary Condition 1:} &\quad v_r(R,\theta) = 0 \\ \text{Boundary Condition 2:} &\quad v_\theta(R,\theta) = 0 \end{aligned}$$

Another two boundary conditions are obtained from the fact that the flow is symmetric about the line which is collinear with the x-axis.

$$\begin{aligned} \text{Boundary Condition 3:} &\quad \frac{\partial v_r}{\partial \theta}(r,0) = 0 \\ \text{Boundary Condition 4:} &\quad \frac{\partial v_r}{\partial \theta}(r,\pi) = 0 \end{aligned}$$

Another two boundary conditions are obtained from the fact that the flow is entirely radial at $\theta = 0$ and $\theta = \pi$.

$$\begin{aligned} \text{Boundary Condition 5:} &\quad v_\theta(r,0) = 0 \\ \text{Boundary Condition 6:} &\quad v_\theta(r,\pi) = 0 \end{aligned}$$

The equation of continuity results by considering a mass balance over a volume element that the fluid is flowing through. Because the fluid is incompressible, the equation simplifies to

$$\nabla \cdot \mathbf{v} = 0.$$

Expand the left side in cylindrical coordinates, using the formula in Appendix B.4 on page 846.

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \underbrace{\frac{\partial v_z}{\partial z}}_{=\,0} = 0$$

Multiply both sides by r.

$$\frac{\partial}{\partial r}(rv_r) + \frac{\partial v_\theta}{\partial \theta} = 0 \tag{1}$$

The equation of motion results by considering a momentum balance over a volume element that the fluid is flowing through. Assuming the fluid viscosity μ and density ρ are constant, the equation simplifies to the Navier-Stokes equation.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla\cdot\rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}$$

As this is a vector equation, it actually represents three scalar equations, one for each variable in the chosen coordinate system. From Appendix B.6 on page 848, the Navier-Stokes equation yields the following three scalar equations in cylindrical coordinates.

$$\rho\Bigg(\underbrace{\frac{\partial v_r}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_r}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_r}{\partial z}}_{=0} - \underbrace{\frac{v_\theta^2}{r}}_{=0}\Bigg) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \underbrace{\frac{\partial^2 v_r}{\partial z^2}}_{=0} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] + \rho g_r$$

$$\rho\Bigg(\underbrace{\frac{\partial v_\theta}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_\theta}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_\theta}{\partial z}}_{=0} + \underbrace{\frac{v_r v_\theta}{r}}_{=0}\Bigg) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \underbrace{\frac{\partial^2 v_\theta}{\partial z^2}}_{=0} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right] + \rho g_\theta$$

$$\rho\Bigg(\underbrace{\frac{\partial v_z}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_z}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_z}{\partial z}}_{=0}\Bigg) = \underbrace{-\frac{\partial p}{\partial z}}_{=0} + \mu\Bigg[\underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)}_{=0} + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_z}{\partial z^2}}_{=0}\Bigg] + \underbrace{\rho g_z}_{=0}$$

Note that all acceleration terms on each left side are neglected because of the creeping flow assumption. The two relevant equations are as follows.

$$0 = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] + \rho g_r \tag{2}$$

$$0 = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right] + \rho g_\theta \tag{3}$$

The system of equations (1), (2), and (3) can be solved for the three unknowns, p, v_r, and v_θ. Multiply both sides of equation (3) by $-r$.

$$0 = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] + \rho g_r$$

$$0 = \frac{\partial p}{\partial \theta} + \mu\left[-r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) - \frac{1}{r}\frac{\partial^2 v_\theta}{\partial \theta^2} - \frac{2}{r}\frac{\partial v_r}{\partial \theta}\right] - \rho g_\theta r$$

In this problem gravity points straight down: $\mathbf{g} = -g\hat{\mathbf{y}}$. Write this unit vector in terms of $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ by using formula A.6-14 on page 827.

$$\mathbf{g} = -g[(\sin\theta)\hat{\mathbf{r}} + (\cos\theta)\hat{\boldsymbol{\theta}}] = -g(\sin\theta)\hat{\mathbf{r}} - g(\cos\theta)\hat{\boldsymbol{\theta}}$$

We see that $g_r = -g\sin\theta$ and $g_\theta = -g\cos\theta$. The previous two equations become

$$0 = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] - \rho g\sin\theta$$

$$0 = \frac{\partial p}{\partial \theta} + \mu\left[-r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) - \frac{1}{r}\frac{\partial^2 v_\theta}{\partial \theta^2} - \frac{2}{r}\frac{\partial v_r}{\partial \theta}\right] + \rho gr\cos\theta.$$

Combine the first and last terms on the right side of each equation.

$$\begin{aligned} 0 &= -\frac{\partial}{\partial r}(p + \rho g r \sin\theta) + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] \\ 0 &= \frac{\partial}{\partial \theta}(p + \rho g r \sin\theta) + \mu\left[-r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) - \frac{1}{r}\frac{\partial^2 v_\theta}{\partial \theta^2} - \frac{2}{r}\frac{\partial v_r}{\partial \theta}\right] \end{aligned}$$

Introduce the modified pressure function $\mathscr{P}(r,\theta) = p(r,\theta) + \rho g r \sin\theta$.

$$0 = -\frac{\partial \mathscr{P}}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] \tag{4}$$

$$0 = \frac{\partial \mathscr{P}}{\partial \theta} + \mu\left[-r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) - \frac{1}{r}\frac{\partial^2 v_\theta}{\partial \theta^2} - \frac{2}{r}\frac{\partial v_r}{\partial \theta}\right] \tag{5}$$

Differentiate both sides of the first equation with respect to θ, and differentiate both sides of the second equation with respect to r.

$$\begin{aligned} 0 &= -\frac{\partial^2 \mathscr{P}}{\partial \theta \partial r} + \mu\frac{\partial}{\partial \theta}\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] \\ 0 &= \frac{\partial^2 \mathscr{P}}{\partial r \partial \theta} + \mu\frac{\partial}{\partial r}\left[-r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) - \frac{1}{r}\frac{\partial^2 v_\theta}{\partial \theta^2} - \frac{2}{r}\frac{\partial v_r}{\partial \theta}\right] \end{aligned}$$

Add the respective sides of each equation in order to eliminate the modified pressure. The mixed derivatives are equal by Clairaut's theorem.

$$0 = \mu\frac{\partial}{\partial \theta}\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] + \mu\frac{\partial}{\partial r}\left[-r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) - \frac{1}{r}\frac{\partial^2 v_\theta}{\partial \theta^2} - \frac{2}{r}\frac{\partial v_r}{\partial \theta}\right]$$

Divide both sides by μ.

$$0 = \frac{\partial}{\partial \theta}\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] + \frac{\partial}{\partial r}\left[-r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) - \frac{1}{r}\frac{\partial^2 v_\theta}{\partial \theta^2} - \frac{2}{r}\frac{\partial v_r}{\partial \theta}\right] \tag{6}$$

Equations (1) and (6) together form a system of two equations for v_r and v_θ. Since both PDEs are linear and homogeneous, the method of separation of variables can be applied to get a solution. Assume that v_r and v_θ have product solutions like so: $v_r = Q(r)T(\theta)$ and $v_\theta = \xi(r)\Theta(\theta)$. In particular, based on boundary conditions 3 and 4, we hypothesize that $T(\theta) = \cos\theta$; in addition, based on boundary conditions 5 and 6, we hypothesize that $\Theta(\theta) = \sin\theta$.

$$\begin{aligned} v_r(r,\theta) &= Q(r)\cos\theta \\ v_\theta(r,\theta) &= \xi(r)\sin\theta \end{aligned}$$

Substitute these formulas into equations (1) and (6).

$$\frac{\partial}{\partial r}[rQ(r)\cos\theta] + \frac{\partial}{\partial \theta}[\xi(r)\sin\theta] = 0$$

$$\begin{aligned} &\frac{\partial}{\partial \theta}\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}[rQ(r)\cos\theta]\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}[Q(r)\cos\theta] - \frac{2}{r^2}\frac{\partial}{\partial \theta}[\xi(r)\sin\theta]\right] \\ &\qquad + \frac{\partial}{\partial r}\left[-r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}[r\xi(r)\sin\theta]\right) - \frac{1}{r}\frac{\partial^2}{\partial \theta^2}[\xi(r)\sin\theta] - \frac{2}{r}\frac{\partial}{\partial \theta}[Q(r)\cos\theta]\right] = 0 \end{aligned}$$

Evaluate the derivatives and expand the left sides.

$$Q'(r)r\cos\theta + Q(r)\cos\theta + \xi(r)\cos\theta = 0$$
$$-Q''(r)\sin\theta + Q'(r)\frac{\sin\theta}{r} - \xi'''(r)r\sin\theta - 2\xi''(r)\sin\theta + \frac{2\sin\theta}{r}\xi'(r) = 0$$

Divide both sides of the first equation by $\cos\theta$, and divide both sides of the second equation by $\sin\theta$.

$$Q'(r)r + Q(r) + \xi(r) = 0$$
$$-Q''(r) + Q'(r)\frac{1}{r} - \xi'''(r)r - 2\xi''(r) + \frac{2}{r}\xi'(r) = 0$$

Solve the first equation for $\xi(r)$

$$\xi(r) = -Q'(r)r - Q(r) \tag{7}$$

and then substitute it into the second equation.

$$-Q''(r) + Q'(r)\frac{1}{r} - r[-Q'(r)r - Q(r)]''' - 2[-Q'(r)r - Q(r)]'' + \frac{2}{r}[-Q'(r)r - Q(r)]' = 0$$

Evaluate the derivatives and expand the left side.

$$r^2Q'''' + 6rQ''' + 3Q'' - \frac{3}{r}Q' = 0$$

Multiply both sides by r^2.

$$r^4Q'''' + 6r^3Q''' + 3r^2Q'' - 3rQ' = 0$$

This is a homogeneous equidimensional ODE, so its solution is of the form $Q = r^m$.

$$Q = r^m \quad \rightarrow \quad Q' = mr^{m-1} \quad \rightarrow \quad Q'' = m(m-1)r^{m-2} \quad \rightarrow \quad Q''' = m(m-1)(m-2)r^{m-3}$$
$$\rightarrow \quad Q'''' = m(m-1)(m-2)(m-3)r^{m-4}$$

Substitute these formulas into the ODE.

$$r^4m(m-1)(m-2)(m-3)r^{m-4} + 6r^3m(m-1)(m-2)r^{m-3} + 3r^2m(m-1)r^{m-2} - 3rmr^{m-1} = 0$$
$$m(m-1)(m-2)(m-3)r^m + 6m(m-1)(m-2)r^m + 3m(m-1)r^m - 3mr^m = 0$$

Divide both sides by r^m.

$$m(m-1)(m-2)(m-3) + 6m(m-1)(m-2) + 3m(m-1) - 3m = 0$$

Expand the left side.

$$m^4 - 4m^2 = 0$$
$$m^2(m^2 - 4) = 0$$
$$m = \{-2, 0, 2\}$$

Three solutions to the ODE are $Q = r^{-2}$ and $Q = r^0 = 1$ and $Q = r^2$. Because the multiplicity of the $m = 0$ root is 2, a second linearly independent solution can be obtained from the first by including a factor of $\ln r$: $Q = r^0 \ln r = \ln r$. By the principle of superposition, the general solution to the ODE is a linear combination of these four.

$$Q(r) = C_1 r^{-2} + C_2 r^2 + C_3 + C_4 \ln r$$

Substitute this formula for Q into equation (7) to get ξ.

$$\xi(r) = \frac{C_1}{r^2} - 3C_2 r^2 - C_3 - C_4(1 + \ln r)$$

Therefore, since $v_r(r, \theta) = Q(r)\cos\theta$ and $v_\theta(r, \theta) = \xi(r)\sin\theta$,

$$v_r(r, \theta) = \left(\frac{C_1}{r^2} + C_2 r^2 + C_3 + C_4 \ln r\right)\cos\theta$$
$$v_\theta(r, \theta) = \left[\frac{C_1}{r^2} - 3C_2 r^2 - C_3 - C_4(1 + \ln r)\right]\sin\theta.$$

Now plug these formulas into equations (4) and (5) to get the modified pressure.

$$0 = -\frac{\partial \mathscr{P}}{\partial r} + 2\mu\left(\frac{C_4}{r^2} + 4C_2\right)\cos\theta$$
$$0 = \frac{\partial \mathscr{P}}{\partial \theta} - 2\mu\left(\frac{C_4}{r} - 4C_2 r\right)\sin\theta$$

Solve for the derivatives.

$$\frac{\partial \mathscr{P}}{\partial r} = 2\mu\left(\frac{C_4}{r^2} + 4C_2\right)\cos\theta$$
$$\frac{\partial \mathscr{P}}{\partial \theta} = 2\mu\left(\frac{C_4}{r} - 4C_2 r\right)\sin\theta$$

Integrate both sides of the first equation partially with respect to r to get $\mathscr{P}$.

$$\mathscr{P}(r, \theta) = 2\mu\left(-\frac{C_4}{r} + 4C_2 r\right)\cos\theta + f(\theta)$$

Differentiate both sides with respect to θ.

$$\frac{\partial \mathscr{P}}{\partial \theta} = 2\mu\left(\frac{C_4}{r} - 4C_2 r\right)\sin\theta + f'(\theta)$$

Comparing this to the previous equation for $\partial \mathscr{P}/\partial\theta$, we see that

$$f'(\theta) = 0.$$

Integrate both sides with respect to θ.

$$f(\theta) = \mathscr{P}_\infty$$

The modified pressure is then

$$\mathscr{P}(r, \theta) = 2\mu\left(-\frac{C_4}{r} + 4C_2 r\right)\cos\theta + \mathscr{P}_\infty.$$

We require that $\mathscr{P} = \mathscr{P}_\infty$ in the limit as $r \to \infty$: $C_2 = 0$.

$$\mathscr{P}(r,\theta) = -2\mu \frac{C_4}{r}\cos\theta + \mathscr{P}_\infty$$

Therefore, since $p(r,\theta) = \mathscr{P}(r,\theta) - \rho g r \sin\theta$,

$$\boxed{p(r,\theta) = \mathscr{P}_\infty - 2\mu \frac{C_4}{r}\cos\theta - \rho g r \sin\theta.}$$

Since $C_2 = 0$, the velocity components become

$$v_r(r,\theta) = \left(\frac{C_1}{r^2} + C_3 + C_4 \ln r\right)\cos\theta$$
$$v_\theta(r,\theta) = \left[\frac{C_1}{r^2} - C_3 - C_4(1 + \ln r)\right]\sin\theta.$$

Apply boundary conditions 1 and 2 to determine C_1 and C_3.

$$v_r(R,\theta) = \left(\frac{C_1}{R^2} + C_3 + C_4 \ln R\right)\cos\theta = 0$$
$$v_\theta(R,\theta) = \left[\frac{C_1}{R^2} - C_3 - C_4(1 + \ln R)\right]\sin\theta = 0$$

Solving this system of equations yields

$$C_1 = \frac{1}{2}C_4 R^2 \quad \text{and} \quad C_3 = -\frac{C_4}{2} - C_4 \ln R.$$

The velocity components become

$$v_r(r,\theta) = \left(\frac{1}{2}\frac{R^2}{r^2}C_4 - \frac{C_4}{2} - C_4 \ln R + C_4 \ln r\right)\cos\theta$$
$$v_\theta(r,\theta) = \left(\frac{1}{2}\frac{R^2}{r^2}C_4 + \frac{C_4}{2} + C_4 \ln R - C_4 - C_4 \ln r\right)\sin\theta,$$

or after simplifying,

$$\boxed{\begin{aligned} v_r(r,\theta) &= 2C_4\left[\frac{1}{2}\ln\left(\frac{r}{R}\right) - \frac{1}{4} + \frac{1}{4}\left(\frac{R}{r}\right)^2\right]\cos\theta \\ v_\theta(r,\theta) &= -2C_4\left[\frac{1}{2}\ln\left(\frac{r}{R}\right) + \frac{1}{4} - \frac{1}{4}\left(\frac{R}{r}\right)^2\right]\sin\theta. \end{aligned}}$$

The formulas in the problem statement are obtained by setting $\mathscr{P}_\infty = p_\infty$ and using one final boundary condition to determine that $2C_4 = Cv_\infty$.