Problem 3C.1

Parallel-disk compression viscometer[6] (Fig. 3C.1). A fluid fills completely the region between two circular disks of radius R. The bottom disk is fixed, and the upper disk is made to approach the lower one very slowly with a constant speed v_0, starting from a height H_0 (and $H_0 \ll R$). The instantaneous height of the upper disk is $H(t)$. It is desired to find the force needed to maintain the speed v_0.

This problem is inherently a rather complicated unsteady-state flow problem. However, a useful approximate solution can be obtained by making two simplifications in the equations of change: (i) we assume that the speed v_0 is so slow that all terms containing time derivatives can be omitted; this is the so-called "quasi-steady-state" assumption; (ii) we use the fact that $H_0 \ll R$ to neglect quite a few terms in the equations of change by order-of-magnitude arguments.

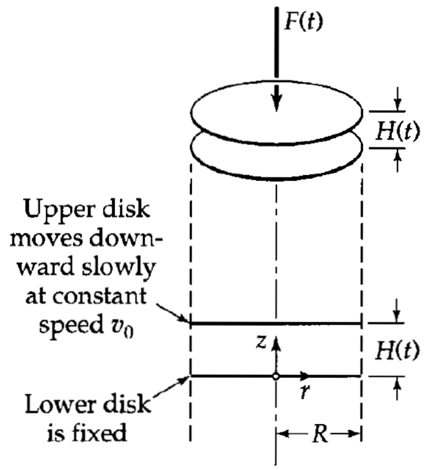


Fig. 3C.1. Squeezing flow in a parallel-disk compression viscometer.

Note that the rate of decrease of the fluid volume between the disks is $\pi R^2 v_0$, and that this must equal the rate of outflow from between the disks, which is $2\pi RH\langle v_r\rangle|_{r=R}$. Hence

$$\langle v_r\rangle|_{r=R} = \frac{Rv_0}{2H(t)} \tag{3C.1-1}$$

We now argue that $v_r(r, z)$ will be of the order of magnitude of $\langle v_r\rangle|_{r=R}$ and that $v_z(r, z)$ is of the order of magnitude of v_0, so that

$$v_r \approx (R/H)v_0; \quad v_z \approx -v_0 \tag{3C.1-2, 3}$$

and hence $|v_z| \ll v_r$. We may now estimate the order of magnitude of various derivatives as follows: as r goes from 0 to R, the radial velocity v_r goes from zero to approximately $(R/H)v_0$. By this kind of reasoning we get

$$\frac{\partial v_r}{\partial r} \approx \frac{(R/H)v_0 - 0}{R - 0} = \frac{v_0}{H} \tag{3C.1-4}$$

$$\frac{\partial v_z}{\partial z} \approx \frac{(-v_0) - 0}{H - 0} = -\frac{v_0}{H}, \text{ etc.} \tag{3C.1-5}$$

[6] J. R. Van Wazer, J. W. Lyons, K. Y. Kim, and R. E. Colwell, *Viscosity and Flow Measurement*, Wiley-Interscience, New York (1963), pp. 292–295.

(a) By the above-outlined order-of-magnitude analysis, show that the continuity equation and the r-component of the equation of motion become (with g_z neglected)

$$\text{continuity:} \qquad \frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{\partial v_z}{\partial z} = 0 \tag{3C.1-6}$$

$$\text{motion:} \qquad 0 = -\frac{dp}{dr} + \mu\frac{\partial^2 v_r}{\partial z^2} \tag{3C.1-7}$$

with the boundary conditions

$$\text{B.C. 1:} \quad \text{at } z = 0, \quad v_r = 0, \quad v_z = 0 \tag{3C.1-8}$$

$$\text{B.C. 2:} \quad \text{at } z = H(t), \quad v_r = 0, \quad v_z = -v_0 \tag{3C.1-9}$$

$$\text{B.C. 3:} \quad \text{at } r = R, \quad p = p_{\text{atm}} \tag{3C.1-10}$$

(b) From Eqs. 3C.1-7 to 9 obtain

$$v_r = \frac{1}{2\mu}\left(\frac{dp}{dr}\right) z(z - H) \tag{3C.1-11}$$

(c) Integrate Eq. 3C.1-6 with respect to z and substitute the result from Eq. 3.C.1-11 to get

$$v_0 = -\frac{H^3}{12\mu}\frac{1}{r}\frac{d}{dr}\left(r\frac{dp}{dr}\right) \tag{3C.1-12}$$

(d) Solve Eq. 3C.1-12 to get the pressure distribution

$$p = p_{\text{atm}} + \frac{3\mu v_0 R^2}{H^3}\left[1 - \left(\frac{r}{R}\right)^2\right] \tag{3C.1-13}$$

(e) Integrate $[(p + \tau_{zz}) - p_{\text{atm}}]$ over the moving-disk surface to find the total force needed to maintain the disk motion:

$$F(t) = \frac{3\pi\mu v_0 R^4}{2[H(t)]^3} \tag{3C.1-14}$$

This result can be used to obtain the viscosity from the force and velocity measurements.

(f) Repeat the analysis for a viscometer that is operated in such a way that a centered, circular glob of liquid never completely fills the space between the two plates. Let the volume of the sample be V and obtain

$$F(t) = \frac{3\mu v_0 V^2}{2\pi[H(t)]^5} \tag{3C.1-15}$$

(g) Repeat the analysis for a viscometer that is operated with constant applied force, F_0. The viscosity is then to be determined by measuring H as a function of time, and the upper-plate velocity is not a constant. Show that

$$\frac{1}{[H(t)]^2} = \frac{1}{H_0^2} + \frac{4F_0 t}{3\pi\mu R^4} \tag{3C.1-16}$$

[TYPO: There should be a colon after "motion" in Eq. 3C.1-7 to be consistent. Also, in part (c) there should not be a period between "3" and "C" in "3.C.1-11."]

Solution

Part (a)

For two concentric disks, one of which moves axially toward the other, we assume that the fluid being squished moves radially and vertically. Both components of velocity vary as a function of r and z.

$$\mathbf{v} = v_r(r,z)\hat{\mathbf{r}} + v_z(r,z)\hat{\mathbf{z}}$$

If we assume the fluid does not slip on the walls, then it has the wall's velocity at $z = 0$ and $z = H = H(t)$.

Boundary Condition 1:	$v_r(r,0) = 0$	Boundary Condition 3:	$v_z(r,0) = 0$
Boundary Condition 2:	$v_r(r,H) = 0$	Boundary Condition 4:	$v_z(r,H) = -v_0$

Since $H(t)$ is very small in comparison to the radius of the disks, gravity (in the negative z-direction) has a negligible influence on the motion. In addition to the fact that v_0 is very small, the pressure can be assumed to be entirely radial: $p = p(r)$. The fluid is in contact with the air at $r = R$, so the pressure there is p_{atm}.

$$\text{Boundary Condition 5:} \quad p(R) = p_{\text{atm}}$$

The equation of continuity results by considering a mass balance over a volume element that the fluid is flowing through. Assuming the fluid density ρ is constant, the equation simplifies to

$$\nabla \cdot \mathbf{v} = 0. \tag{1}$$

The equation of motion results by considering a momentum balance over a volume element that the fluid is flowing through. Assuming the fluid viscosity μ is also constant, the equation simplifies to the Navier-Stokes equation.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla \cdot \rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g} \tag{2}$$

As this is a vector equation, it actually represents three scalar equations—one for each variable in the chosen coordinate system. Using cylindrical coordinates is the appropriate choice for this problem, so equations (1) and (2) will be used in (r, θ, z). From Appendix B.4 on page 846, the continuity equation becomes

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \underbrace{\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}}_{=\,0} + \frac{\partial v_z}{\partial z} = 0.$$

From Appendix B.6 on page 848, the Navier-Stokes equation yields the following three scalar equations in cylindrical coordinates.

$$\rho\left(\underbrace{\frac{\partial v_r}{\partial t}}_{=\,0} + v_r\frac{\partial v_r}{\partial r} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}}_{=\,0} + v_z\frac{\partial v_r}{\partial z} - \underbrace{\frac{v_\theta^2}{r}}_{=\,0}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right) + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}}_{=\,0} + \frac{\partial^2 v_r}{\partial z^2} - \underbrace{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}}_{=\,0}\right] + \underbrace{\rho g_r}_{=\,0}$$

$$\rho\left(\underbrace{\frac{\partial v_\theta}{\partial t}}_{=\,0} + \underbrace{v_r\frac{\partial v_\theta}{\partial r}}_{=\,0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}}_{=\,0} + \underbrace{v_z\frac{\partial v_\theta}{\partial z}}_{=\,0} + \underbrace{\frac{v_r v_\theta}{r}}_{=\,0}\right) = \underbrace{-\frac{1}{r}\frac{\partial p}{\partial \theta}}_{=\,0} + \mu\left[\underbrace{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right)}_{=\,0} + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}}_{=\,0} + \underbrace{\frac{\partial^2 v_\theta}{\partial z^2}}_{=\,0} + \underbrace{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}}_{=\,0}\right] + \underbrace{\rho g_\theta}_{=\,0}$$

$$\rho\left(\underbrace{\frac{\partial v_z}{\partial t}}_{=\,0} + v_r\frac{\partial v_z}{\partial r} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}}_{=\,0} + v_z\frac{\partial v_z}{\partial z}\right) = \underbrace{-\frac{\partial p}{\partial z}}_{=\,0} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}}_{=\,0} + \frac{\partial^2 v_z}{\partial z^2}\right] + \underbrace{\rho g_z}_{=\,0}$$

Consider the r-equation.

$$\rho\left(v_r\frac{\partial v_r}{\partial r}+v_z\frac{\partial v_r}{\partial z}\right)=-\frac{dp}{dr}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)+\frac{\partial^2 v_r}{\partial z^2}\right]$$

It will be approximated by replacing each of the first derivatives with its corresponding difference quotient.

$$\begin{aligned}\frac{\partial v_r}{\partial r}&\approx\frac{v_r(R,z)-v_r(0,z)}{R-0}=\frac{(Rv_0/H)-0}{R}=\frac{v_0}{H}\\ \frac{\partial v_r}{\partial z}&\approx\frac{v_r(r,H)-v_r(r,0)}{H-0}=\frac{0-0}{H}=0\\ \frac{\partial}{\partial r}(rv_r)&\approx\frac{Rv_r(R,z)-0v_r(0,z)}{R-0}=v_r(R,z)=\frac{Rv_0}{H}\end{aligned}$$

Substitute these formulas along with $v_r\approx(R/H)v_0$ into the r-equation.

$$\rho\left(\frac{Rv_0}{H}\cdot\frac{v_0}{H}+v_z\cdot 0\right)=-\frac{dp}{dr}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{Rv_0}{H}\right)+\frac{\partial^2 v_r}{\partial z^2}\right]$$

$$\rho\left(\frac{Rv_0^2}{H^2}\right)=-\frac{dp}{dr}+\mu\left[-\frac{1}{r^2}\left(\frac{Rv_0}{H}\right)+\frac{\partial^2 v_r}{\partial z^2}\right]$$

The desired equation of motion follows by requiring that

$$\begin{aligned}\frac{Rv_0^2}{H^2}&\ll 1 \qquad\text{and}\qquad \frac{Rv_0}{H}\ll 1\\ v_0&\ll\frac{H}{\sqrt{R}} \qquad\text{and}\qquad v_0\ll\frac{H}{R},\end{aligned}$$

which can be combined as

$$v_0\ll\frac{H}{R}.$$

The left side will be nearly zero, and the first term in square brackets will be negligible compared to the second term. The continuity equation and the approximated r-equation are therefore

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r)+\frac{\partial v_z}{\partial z}=0 \tag{3}$$

$$0=-\frac{dp}{dr}+\mu\frac{\partial^2 v_r}{\partial z^2}. \tag{4}$$

Part (b)

Solve equation (4) for $\partial^2 v_r/\partial z^2$.

$$\frac{\partial^2 v_r}{\partial z^2}=\frac{1}{\mu}\frac{dp}{dr}$$

Integrate both sides partially with respect to z.

$$\frac{\partial v_r}{\partial z}=\frac{1}{\mu}\frac{dp}{dr}z+f_1(r)$$

Integrate both sides partially with respect to z once more.

$$v_r(r,z)=\frac{1}{2\mu}\frac{dp}{dr}z^2+f_1(r)z+f_2(r)$$

Here $f_1(r)$ and $f_2(r)$ are arbitrary differentiable functions of r. Use the two boundary conditions, $v_r(r,0) = 0$ and $v_r(r,H) = 0$, to determine them.

$$\begin{aligned} v_r(r,0) &= f_2(r) = 0 \\ v_r(r,H) &= \frac{1}{2\mu}\frac{dp}{dr}H^2 + f_1(r)H + f_2(r) = 0 \end{aligned}$$

Solving this system of equations yields

$$f_1(r) = -\frac{1}{2\mu}\frac{dp}{dr}H \quad \text{and} \quad f_2(r) = 0.$$

Therefore,

$$\begin{aligned} v_r(r,z) &= \frac{1}{2\mu}\frac{dp}{dr}z^2 - \frac{1}{2\mu}\frac{dp}{dr}Hz \\ &= \frac{1}{2\mu}\frac{dp}{dr}z(z-H). \end{aligned}$$

Part (c)

Substitute this result for $v_r(r,z)$ into equation (3).

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{1}{2\mu}\frac{dp}{dr}z(z-H)\right] + \frac{\partial v_z}{\partial z} = 0$$

$$\frac{1}{2\mu r}\frac{d}{dr}\left(r\frac{dp}{dr}\right)z(z-H) + \frac{\partial v_z}{\partial z} = 0$$

$$\frac{1}{2\mu r}\frac{d}{dr}\left(r\frac{dp}{dr}\right)(z^2 - Hz) + \frac{\partial v_z}{\partial z} = 0$$

Integrate both sides partially with respect to z.

$$\frac{1}{2\mu r}\frac{d}{dr}\left(r\frac{dp}{dr}\right)\left(\frac{z^3}{3} - H\frac{z^2}{2}\right) + v_z = f_3(r)$$

Set $z = 0$ and apply the boundary condition $v_z(r,0) = 0$ to determine $f_3(r)$.

$$\frac{1}{2\mu r}\frac{d}{dr}\left(r\frac{dp}{dr}\right)(0-0) + 0 = f_3(r) \quad \rightarrow \quad f_3(r) = 0$$

Consequently,

$$\frac{1}{2\mu r}\frac{d}{dr}\left(r\frac{dp}{dr}\right)\left(\frac{z^3}{3} - H\frac{z^2}{2}\right) + v_z = 0.$$

Let $z = H$

$$\frac{1}{2\mu r}\frac{d}{dr}\left(r\frac{dp}{dr}\right)\left(\frac{H^3}{3} - H\frac{H^2}{2}\right) + v_z(r,H) = 0$$

and use the boundary condition $v_z(r,H) = -v_0$.

$$\frac{1}{2\mu r}\frac{d}{dr}\left(r\frac{dp}{dr}\right)\left(-\frac{H^3}{6}\right) - v_0 = 0$$

Solving for v_0 then,

$$v_0 = -\frac{H^3}{12\mu}\frac{1}{r}\frac{d}{dr}\left(r\frac{dp}{dr}\right).$$

Part (d)

Multiply both sides by $-12\mu r/H^3$.

$$\frac{d}{dr}\left(r\frac{dp}{dr}\right) = -\frac{12\mu v_0}{H^3}r$$

Integrate both sides with respect to r.

$$r\frac{dp}{dr} = -\frac{6\mu v_0}{H^3}r^2 + C_1$$

Divide both sides by r.

$$\frac{dp}{dr} = -\frac{6\mu v_0}{H^3}r + \frac{C_1}{r}$$

Integrate both sides with respect to r once more.

$$p(r) = -\frac{3\mu v_0}{H^3}r^2 + C_1 \ln r + C_2$$

For the pressure to remain finite as $r \to 0$, set $C_1 = 0$. Apply the boundary condition $p(R) = p_{\text{atm}}$ to determine C_2.

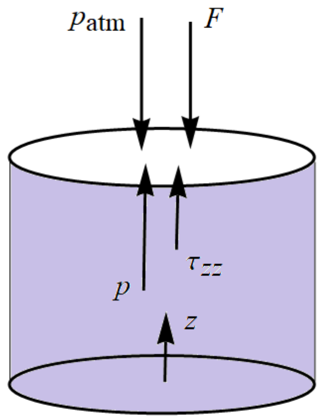
$$p(R) = -\frac{3\mu v_0}{H^3}R^2 + C_2 = p_{\text{atm}} \quad \rightarrow \quad C_2 = p_{\text{atm}} + \frac{3\mu v_0}{H^3}R^2$$

Therefore,

$$\begin{aligned} p(r) &= -\frac{3\mu v_0}{H^3}r^2 + p_{\text{atm}} + \frac{3\mu v_0}{H^3}R^2 \\ &= p_{\text{atm}} + \frac{3\mu v_0}{H^3}(R^2 - r^2) \\ &= p_{\text{atm}} + \frac{3\mu v_0 R^2}{H^3}\left(1 - \frac{r^2}{R^2}\right) \\ &= p_{\text{atm}} + \frac{3\mu v_0 R^2}{H^3}\left[1 - \left(\frac{r}{R}\right)^2\right]. \end{aligned}$$

Part (e)

Draw a free-body diagram of the descending disk (never mind the scaling).



Apply Newton's second law in the z-direction to obtain an equation involving the applied force and the forces due to pressure and viscosity.

$$\sum F_z = ma_z$$

Since we want the force needed to maintain the speed v_0, the acceleration a_z is zero.

$$-F - \int p_{\text{atm}}\, dA + \int p\, dA + \int \tau_{zz}\, dA = 0$$

Solve for F and combine the integrals.

$$F = \int (p - p_{\text{atm}} + \tau_{zz})\, dA$$

Substitute the result found in part (d) for p. From Appendix B.1 on page 844, the viscous force in the z-direction in cylindrical coordinates is

$$\begin{aligned} \tau_{zz} &= -\mu\left[2\frac{\partial v_z}{\partial z}\right] + \left(\frac{2}{3}\mu - \kappa\right)(\nabla \cdot \mathbf{v}) \\ &= -2\mu\frac{\partial v_z}{\partial z} \\ &\approx -2\mu\frac{(-v_0) - 0}{H - 0} \\ &= \frac{2\mu v_0}{H}. \end{aligned}$$

Plugging in the formulas and using $dA = 2\pi r\, dr$, results in

$$\begin{aligned} F &= \int_0^R \left\{\cancel{p_{\text{atm}}} + \frac{3\mu v_0 R^2}{H^3}\left[1 - \left(\frac{r}{R}\right)^2\right] - \cancel{p_{\text{atm}}} + \frac{2\mu v_0}{H}\right\}(2\pi r\, dr) \\ &= \frac{6\pi\mu v_0 R^2}{H^3}\int_0^R \left(r - \frac{r^3}{R^2}\right) dr + \frac{4\pi\mu v_0}{H}\int_0^R r\, dr \\ &= \frac{6\pi\mu v_0 R^2}{H^3}\left(\frac{R^2}{4}\right) + \frac{4\pi\mu v_0}{H}\left(\frac{R^2}{2}\right). \end{aligned}$$

Factor this result.

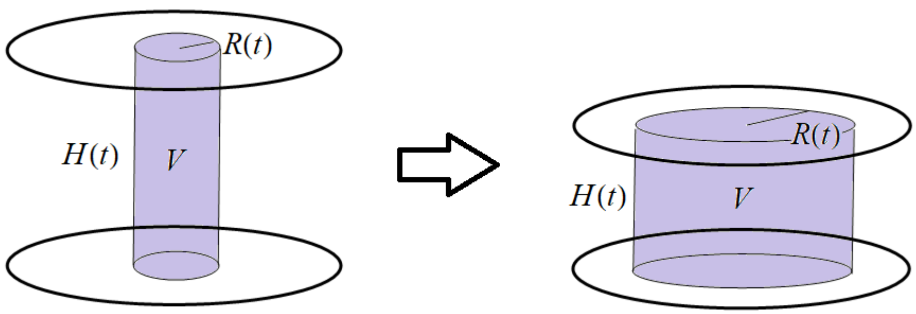
$$\begin{aligned} F &= \frac{3\pi\mu v_0 R^4}{2H^3} + \frac{2\pi\mu v_0 R^2}{H} \\ &= \frac{3\pi\mu v_0 R^4}{2H^3}\Bigg(1 + \underbrace{\frac{4H^2}{3R^2}}_{\approx 0}\Bigg) \end{aligned}$$

This term is approximately zero because $H \ll R$. Therefore,

$$F(t) \approx \frac{3\pi\mu v_0 R^4}{2[H(t)]^3}.$$

Part (f)

In this situation the liquid has the shape of a circular cylinder that doesn't quite fill up the space between the disks.



Nothing changes in the preceding analysis except that now $R = R(t)$. The relationship between $R(t)$, V and $H(t)$ is given by the formula for the volume of a cylinder.

$$V = \pi[R(t)]^2 H(t)$$

Solve for $[R(t)]^2$.

$$[R(t)]^2 = \frac{V}{\pi H(t)}$$

Square both sides.

$$[R(t)]^4 = \frac{V^2}{\pi^2 [H(t)]^2}$$

Substitute this into the result of part (e).

$$F(t) \approx \frac{3\pi\mu v_0}{2[H(t)]^3} \cdot \frac{V^2}{\pi^2 [H(t)]^2}$$

Therefore,

$$F(t) \approx \frac{3\mu v_0 V^2}{2\pi[H(t)]^5}.$$

Part (g)

The applied force is now constant, and the velocity with which the top disk descends is no longer constant. Replace F with F_0 and boundary condition 4, $v_z(r, H) = -v_0$, with

$$\text{Boundary Condition 4:} \quad v_z(r, H) = \frac{dH}{dt}.$$

Doing so does not affect any of the analysis in parts (a) through (e). Begin with the result of part (e) with the replacements.

$$F(t) = \frac{3\pi\mu v_0 R^4}{2[H(t)]^3} \quad \rightarrow \quad F_0 = \frac{3\pi\mu\left(-\frac{dH}{dt}\right)R^4}{2[H(t)]^3}$$

This equation for the force has turned into a separable ODE for the height. Separate variables.

$$-\frac{2F_0}{3\pi\mu R^4}\,dt = \frac{dH}{H^3}$$

Integrate both sides.

$$\int -\frac{2F_0}{3\pi\mu R^4}\,dt = \int \frac{dH}{H^3}$$

$$-\frac{2F_0}{3\pi\mu R^4}t + C_3 = -\frac{1}{2H^2}$$

Apply the initial condition $H(0) = H_0$ to determine C_3.

$$C_3 = -\frac{1}{2H_0^2}$$

Plug this result into the previous equation.

$$-\frac{2F_0}{3\pi\mu R^4}t - \frac{1}{2H_0^2} = -\frac{1}{2H^2}$$

Therefore, multiplying both sides by -2,

$$\frac{1}{[H(t)]^2} = \frac{1}{H_0^2} + \frac{4F_0 t}{3\pi\mu R^4}.$$