Problem 3C.2

Normal stresses at solid surfaces for compressible fluids. Extend Example 3.1-1 to compressible fluids. Show that

$$\tau_{zz}|_{z=0} = \left(\frac{4}{3}\mu + \kappa\right)(\partial \ln \rho/\partial t)\bigg|_{z=0} \tag{3C.2-1}$$

Discuss the physical significance of this result.

Solution

Suppose there exists a solid object of arbitrary shape and that a compressible fluid is flowing around it with velocity $\mathbf{v}$. Let P be a point on the object's surface and set this to be the origin of a Cartesian coordinate system, choosing the z-axis to be perpendicular to the surface. From Appendix B.1 on page 843, the normal stress to the surface is given by

$$\begin{aligned}\tau_{zz} &= -\mu\left[2\frac{\partial v_z}{\partial z}\right] + \left(\frac{2}{3}\mu - \kappa\right)(\nabla \cdot \mathbf{v}) \\ &= 2\mu\left(-\frac{\partial v_z}{\partial z}\right) + \left(\frac{2}{3}\mu - \kappa\right)\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right). \end{aligned} \tag{1}$$

The equation of continuity results by considering a mass balance over a volume element that the fluid is flowing through.

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= -(\nabla \cdot \rho\mathbf{v}) \\ &= -\left(\frac{\partial}{\partial x}\rho v_x + \frac{\partial}{\partial y}\rho v_y + \frac{\partial}{\partial z}\rho v_z\right)\end{aligned}$$

Bring all terms to the left side.

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\rho v_x + \frac{\partial}{\partial y}\rho v_y + \frac{\partial}{\partial z}\rho v_z = 0$$

Expand the left side by using the product rule.

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}v_x + \frac{\partial \rho}{\partial y}v_y + \frac{\partial \rho}{\partial z}v_z + \rho\frac{\partial v_x}{\partial x} + \rho\frac{\partial v_y}{\partial y} + \rho\frac{\partial v_z}{\partial z} = 0$$

Solve this equation for $\partial v_x/\partial x + \partial v_y/\partial y + \partial v_z/\partial z$ and for $-\partial v_z/\partial z$.

$$\begin{cases} \dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} + \dfrac{\partial v_z}{\partial z} = -\dfrac{1}{\rho}\dfrac{\partial \rho}{\partial t} - \dfrac{1}{\rho}\dfrac{\partial \rho}{\partial x}v_x - \dfrac{1}{\rho}\dfrac{\partial \rho}{\partial y}v_y - \dfrac{1}{\rho}\dfrac{\partial \rho}{\partial z}v_z \\ -\dfrac{\partial v_z}{\partial z} = \dfrac{1}{\rho}\dfrac{\partial \rho}{\partial t} + \dfrac{1}{\rho}\dfrac{\partial \rho}{\partial x}v_x + \dfrac{1}{\rho}\dfrac{\partial \rho}{\partial y}v_y + \dfrac{1}{\rho}\dfrac{\partial \rho}{\partial z}v_z + \dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} \end{cases}$$

Substitute these results into equation (1).

$$\begin{aligned}\tau_{zz} = 2\mu&\left(\frac{1}{\rho}\frac{\partial \rho}{\partial t} + \frac{1}{\rho}\frac{\partial \rho}{\partial x}v_x + \frac{1}{\rho}\frac{\partial \rho}{\partial y}v_y + \frac{1}{\rho}\frac{\partial \rho}{\partial z}v_z + \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right) \\ &+ \left(\frac{2}{3}\mu - \kappa\right)\left(-\frac{1}{\rho}\frac{\partial \rho}{\partial t} - \frac{1}{\rho}\frac{\partial \rho}{\partial x}v_x - \frac{1}{\rho}\frac{\partial \rho}{\partial y}v_y - \frac{1}{\rho}\frac{\partial \rho}{\partial z}v_z\right)\end{aligned}$$

Now evaluate τ_{zz} at $z = 0$. Assuming that the fluid does not slip on the object's surface, we have

$$v_x = v_y = v_z = \frac{\partial v_x}{\partial x} = \frac{\partial v_y}{\partial y} = 0 \quad \text{at } z = 0.$$

Therefore,

$$\begin{aligned}
\tau_{zz}|_{z=0} &= 2\mu \left(\frac{1}{\rho}\frac{\partial \rho}{\partial t} + \frac{1}{\rho}\frac{\partial \rho}{\partial x}v_x + \frac{1}{\rho}\frac{\partial \rho}{\partial y}v_y + \frac{1}{\rho}\frac{\partial \rho}{\partial z}v_z + \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right)\Bigg|_{z=0} \\
&\qquad + \left(\frac{2}{3}\mu - \kappa\right)\left(-\frac{1}{\rho}\frac{\partial \rho}{\partial t} - \frac{1}{\rho}\frac{\partial \rho}{\partial x}v_x - \frac{1}{\rho}\frac{\partial \rho}{\partial y}v_y - \frac{1}{\rho}\frac{\partial \rho}{\partial z}v_z \right)\Bigg|_{z=0} \\
&= 2\mu\left(\frac{1}{\rho}\frac{\partial \rho}{\partial t}\right)\Bigg|_{z=0} + \left(\frac{2}{3}\mu - \kappa\right)\left(-\frac{1}{\rho}\frac{\partial \rho}{\partial t}\right)\Bigg|_{z=0} \\
&= \left(2\mu - \frac{2}{3}\mu + \kappa\right)\left(\frac{1}{\rho}\frac{\partial \rho}{\partial t}\right)\Bigg|_{z=0} \\
&= \left(\frac{4}{3}\mu + \kappa\right)\left(\frac{1}{\rho}\frac{\partial \rho}{\partial t}\right)\Bigg|_{z=0} \\
&= \left(\frac{4}{3}\mu + \kappa\right)\left(\frac{\partial}{\partial t}\ln \rho\right)\Bigg|_{z=0},
\end{aligned}$$

where the chain rule was used in this last step. The normal stress on a solid surface is not zero then if the fluid density changes with respect to time, that is, if the compressible flow is unsteady.