Problem 3C.4

Alternative methods of solving the Couette viscometer problem by use of angular momentum concepts (Fig. 3.6-1).

(a) By making a *shell angular-momentum balance* on a thin shell of thickness Δr, show that

$$\frac{d}{dr}(r^2\tau_{r\theta}) = 0 \tag{3C.4-1}$$

Next insert the appropriate expression for $\tau_{r\theta}$ in terms of the gradient of the tangential component of the velocity. Then solve the resulting differential equation with the boundary conditions to get Eq. 3.6-29.

(b) Show how to obtain Eq. 3C.4-1 from the *equation of change for angular momentum* given in Eq. 3.4-1.

Solution

The system to analyze is shown in Fig. 3.6-1 on page 90 of the textbook.

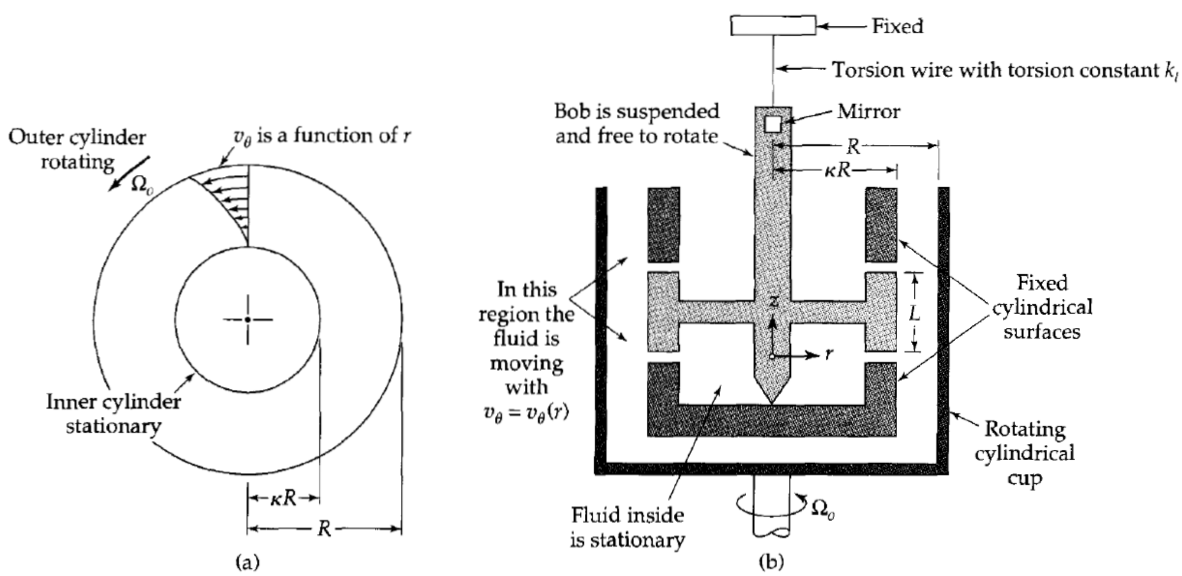


Fig. 3.6-1. (*a*)Tangential laminar flow of an incompressible fluid in the space between two cylinders; the outer one is moving with an angular velocity Ω_o. (*b*) A diagram of a Couette viscometer. One measures the angular velocity Ω_o of the cup and the deflection θ_B of the bob at steady-state operation. Equation 3.6-31 gives the viscosity μ in terms of Ω_o and the torque $T_z = k_t\theta_B$.

On page 41 in §2.1 it was stated that the shell momentum balance method only applies to flows that are rectilinear. The flow in the Couette viscometer is purely rotational, so while the momentum balance over a shell may not be useful, another kind of balance involving angular momentum will be.

Part (a)

Assume that the fluid velocity only has an angular component and that it varies with radius.

$$\mathbf{v} = v_\theta(r)\hat{\boldsymbol{\theta}}$$

Looking at the components of the viscous stress tensor $\boldsymbol{\tau}$ in cylindrical coordinates on page 844, we see that only $\tau_{r\theta}$ and $\tau_{\theta r}$ are nonzero. Consequently, the combined momentum fluxes,

$$\phi_{r\theta} = \tau_{r\theta} + \underbrace{\rho v_r v_\theta}_{=0}$$
$$\phi_{\theta r} = \tau_{\theta r} + \underbrace{\rho v_\theta v_r}_{=0}$$

are relevant. The pressure is assumed to vary with radius and height because of centrifugal and gravitational forces, respectively.

$$p = p(r, z)$$

Consequently, the combined momentum fluxes,

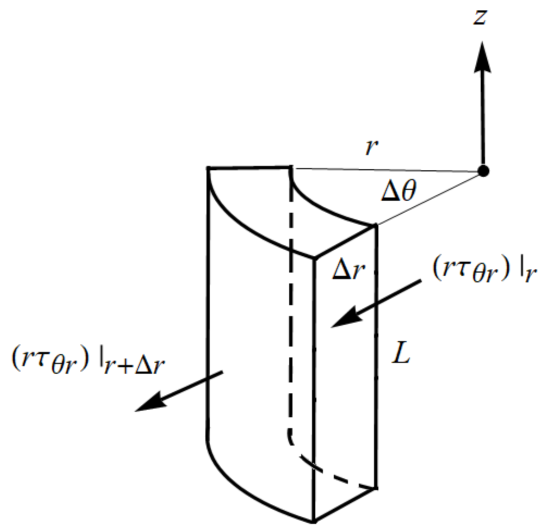
$$\phi_{rr} = p + \underbrace{\tau_{rr}}_{=0} + \underbrace{\rho v_r v_r}_{=0}$$
$$\phi_{zz} = p + \underbrace{\tau_{zz}}_{=0} + \underbrace{\rho v_z v_z}_{=0},$$

are also relevant. Take the cross product of the position vector $\mathbf{r}$ with the combined momentum flux tensor $\boldsymbol{\phi}$ to get the angular momentum flux, a second-order tensor. In cylindrical coordinates the position vector is $\mathbf{r} = r\hat{\mathbf{r}} + z\hat{\mathbf{z}}$.

$$\begin{aligned}
\mathbf{r} \times \boldsymbol{\phi} &= (r\boldsymbol{\delta}_r + z\boldsymbol{\delta}_z) \times (\phi_{rr}\boldsymbol{\delta}_r\boldsymbol{\delta}_r + \phi_{r\theta}\boldsymbol{\delta}_r\boldsymbol{\delta}_\theta + \phi_{rz}\boldsymbol{\delta}_r\boldsymbol{\delta}_z \\
&\qquad\qquad\qquad\quad + \phi_{\theta r}\boldsymbol{\delta}_\theta\boldsymbol{\delta}_r + \phi_{\theta\theta}\boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta + \phi_{\theta z}\boldsymbol{\delta}_\theta\boldsymbol{\delta}_z \\
&\qquad\qquad\qquad\quad + \phi_{zr}\boldsymbol{\delta}_z\boldsymbol{\delta}_r + \phi_{z\theta}\boldsymbol{\delta}_z\boldsymbol{\delta}_\theta + \phi_{zz}\boldsymbol{\delta}_z\boldsymbol{\delta}_z) \\
&= (r\boldsymbol{\delta}_r + z\boldsymbol{\delta}_z) \times (p\boldsymbol{\delta}_r\boldsymbol{\delta}_r + \tau_{r\theta}\boldsymbol{\delta}_r\boldsymbol{\delta}_\theta + \tau_{\theta r}\boldsymbol{\delta}_\theta\boldsymbol{\delta}_r + p\boldsymbol{\delta}_z\boldsymbol{\delta}_z) \\
&= rp(\boldsymbol{\delta}_r \times \boldsymbol{\delta}_r)\boldsymbol{\delta}_r + r\tau_{r\theta}(\boldsymbol{\delta}_r \times \boldsymbol{\delta}_r)\boldsymbol{\delta}_\theta + r\tau_{\theta r}(\boldsymbol{\delta}_r \times \boldsymbol{\delta}_\theta)\boldsymbol{\delta}_r + rp(\boldsymbol{\delta}_r \times \boldsymbol{\delta}_z)\boldsymbol{\delta}_z \\
&\qquad + zp(\boldsymbol{\delta}_z \times \boldsymbol{\delta}_r)\boldsymbol{\delta}_r + z\tau_{r\theta}(\boldsymbol{\delta}_z \times \boldsymbol{\delta}_r)\boldsymbol{\delta}_\theta + z\tau_{\theta r}(\boldsymbol{\delta}_z \times \boldsymbol{\delta}_\theta)\boldsymbol{\delta}_r + zp(\boldsymbol{\delta}_z \times \boldsymbol{\delta}_z)\boldsymbol{\delta}_z \\
&= rp(\mathbf{0})\boldsymbol{\delta}_r + r\tau_{r\theta}(\mathbf{0})\boldsymbol{\delta}_\theta + r\tau_{\theta r}(\boldsymbol{\delta}_z)\boldsymbol{\delta}_r + rp(-\boldsymbol{\delta}_\theta)\boldsymbol{\delta}_z \\
&\qquad + zp(\boldsymbol{\delta}_\theta)\boldsymbol{\delta}_r + z\tau_{r\theta}(\boldsymbol{\delta}_\theta)\boldsymbol{\delta}_\theta + z\tau_{\theta r}(-\boldsymbol{\delta}_r)\boldsymbol{\delta}_r + zp(\mathbf{0})\boldsymbol{\delta}_z \\
&= r\tau_{\theta r}\boldsymbol{\delta}_z\boldsymbol{\delta}_r - rp\boldsymbol{\delta}_\theta\boldsymbol{\delta}_z + zp\boldsymbol{\delta}_\theta\boldsymbol{\delta}_r + z\tau_{r\theta}\boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta - z\tau_{\theta r}\boldsymbol{\delta}_r\boldsymbol{\delta}_r
\end{aligned}$$

The quantity $r\tau_{\theta r}$ represents the z-component of angular momentum (per unit area) that the r-momentum (per unit area) makes. Since it's solely in terms of r, a z-angular momentum balance over a shell that the fluid is flowing through can be used to get an ODE for $\tau_{\theta r}$.

Consider a piece of a cylindrical shell in the annular region with radius r, thickness Δr, and length L.



Write the z-angular momentum balance, multiplying the angular momentum flux by the respective areas, adding what goes into the shell, and subtracting what comes out of the shell.

$$+(r\Delta\theta\, L)(r\tau_{\theta r})|_r - [(r+\Delta r)\Delta\theta\, L](r\tau_{\theta r})|_{r+\Delta r} = 0$$

In this second term, $r+\Delta r$ can be written as r because r is evaluated at $r+\Delta r$.

$$[(r\Delta\theta\, L)(r\tau_{\theta r})]|_r - [(r\Delta\theta\, L)(r\tau_{\theta r})]|_{r+\Delta r} = 0$$

$$[(L\Delta\theta)(r^2\tau_{\theta r})]\big|_r - [(L\Delta\theta)(r^2\tau_{\theta r})]\big|_{r+\Delta r} = 0$$

$$(-L\Delta\theta)\left[(r^2\tau_{\theta r})\big|_{r+\Delta r} - (r^2\tau_{\theta r})\big|_r\right] = 0$$

Divide both sides by $-L\Delta\theta\,\Delta r$.

$$\frac{r^2\tau_{\theta r}\big|_{r+\Delta r} - r^2\tau_{\theta r}\big|_r}{\Delta r} = 0$$

Take the limit of both sides as $\Delta r \to 0$.

$$\lim_{\Delta r\to 0}\frac{r^2\tau_{\theta r}\big|_{r+\Delta r} - r^2\tau_{\theta r}\big|_r}{\Delta r} = \lim_{\Delta r\to 0} 0$$

On the left is how the first derivative of $r^2\tau_{\theta r}$ with respect to r is defined.

$$\frac{d}{dr}(r^2\tau_{\theta r}) = 0$$

Substitute the formula for $\tau_{\theta r}$ in cylindrical coordinates on page 844.

$$\frac{d}{dr}\left\{r^2\left[-\mu r\frac{d}{dr}\left(\frac{v_\theta}{r}\right)\right]\right\} = 0$$

Divide both sides by $-\mu$.

$$\frac{d}{dr}\left[r^3\frac{d}{dr}\left(\frac{v_\theta}{r}\right)\right] = 0$$

Integrate both sides with respect to r.

$$r^3 \frac{d}{dr}\left(\frac{v_\theta}{r}\right) = C_1$$

Divide both sides by r^3.

$$\frac{d}{dr}\left(\frac{v_\theta}{r}\right) = \frac{C_1}{r^3}$$

Integrate both sides with respect to r once more.

$$\frac{v_\theta}{r} = -\frac{C_1}{2r^2} + C_2$$

Multiply both sides by r.

$$v_\theta(r) = -\frac{C_1}{2r} + C_2 r$$

Use a new constant C_3 for $-C_1/2$.

$$v_\theta(r) = \frac{C_3}{r} + C_2 r$$

Since there are two arbitrary constants, two boundary conditions are needed. The inner cylinder is stationary, and the outer cylinder is rotating with angular velocity $\Omega_o\hat{\mathbf{z}}$, which means the boundary conditions for the (tangential) velocity are

$$\begin{aligned} v_\theta(\kappa R) &= 0 \\ v_\theta(R) &= \Omega_o R. \end{aligned}$$

Apply them now to determine C_2 and C_3.

$$\begin{aligned} v_\theta(\kappa R) &= \frac{C_3}{\kappa R} + C_2 \kappa R = 0 \\ v_\theta(R) &= \frac{C_3}{R} + C_2 R = \Omega_o R \end{aligned}$$

Solving this system of equations yields

$$C_2 = \frac{\Omega_o}{1-\kappa^2} \quad \text{and} \quad C_3 = -\frac{\kappa^2 R^2}{1-\kappa^2}\Omega_o.$$

Therefore, Eq. 3.6-29 is obtained.

$$\begin{aligned} v_\theta(r) &= -\frac{\kappa^2 R^2}{1-\kappa^2}\Omega_o \frac{1}{r} + \frac{\Omega_o}{1-\kappa^2} r \\ &= -\frac{\Omega_o R}{\frac{1-\kappa^2}{\kappa}} \frac{\kappa R}{r} + \frac{\Omega_o R}{\frac{1-\kappa^2}{\kappa}} \frac{r}{\kappa R} \\ &= \Omega_o R \frac{\left(\frac{r}{\kappa R} - \frac{\kappa R}{r}\right)}{\left(\frac{1}{\kappa} - \kappa\right)} \end{aligned} \tag{3.6-29}$$

Part (b)

The equation of change for angular momentum will be derived from the equation of motion. Then it will be shown how Eq. 3C.4-1 comes as a result.

$$\frac{\partial}{\partial t}\rho\mathbf{v} = -\nabla\cdot\rho\mathbf{v}\mathbf{v} - \nabla p - \nabla\cdot\boldsymbol{\tau} + \rho\mathbf{g} \tag{3.2-9}$$

Take the cross product of the position vector $\mathbf{r}$ with both sides. Note that $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$ is unrelated to the fluid velocity $\mathbf{v}$, which is a function of x, y, z, and t.

$$\begin{aligned}\mathbf{r}\times\frac{\partial}{\partial t}\rho\mathbf{v} &= \mathbf{r}\times(-\nabla\cdot\rho\mathbf{v}\mathbf{v} - \nabla p - \nabla\cdot\boldsymbol{\tau} + \rho\mathbf{g}) \\ &= -\mathbf{r}\times(\nabla\cdot\rho\mathbf{v}\mathbf{v}) - \mathbf{r}\times(\nabla p) - \mathbf{r}\times(\nabla\cdot\boldsymbol{\tau}) + \mathbf{r}\times(\rho\mathbf{g})\end{aligned}$$

Each of the terms in this equation will be examined one by one.

$$\begin{aligned}\mathbf{r}\times\frac{\partial}{\partial t}\rho\mathbf{v} &= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i x_i\right)\times\frac{\partial}{\partial t}\rho\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j v_j\right) \\ &= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i x_i\right)\times\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial t}\rho v_j\right) \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3}(\boldsymbol{\delta}_i\times\boldsymbol{\delta}_j)x_i\frac{\partial}{\partial t}\rho v_j \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3}(\boldsymbol{\delta}_i\times\boldsymbol{\delta}_j)\frac{\partial}{\partial t}\rho x_i v_j \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_k\varepsilon_{ijk}\frac{\partial}{\partial t}\rho x_i v_j \\ &= \frac{\partial}{\partial t}\rho\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_k\varepsilon_{ijk}x_i v_j\right) \\ &= \frac{\partial}{\partial t}\rho(\mathbf{r}\times\mathbf{v})\end{aligned}$$

$$
\begin{aligned}
\mathbf{r} \times (\nabla \cdot \rho \mathbf{v}\mathbf{v}) &= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i x_i\right) \times \left[\left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial}{\partial x_j}\right) \cdot \rho \left(\sum_{k=1}^{3} \boldsymbol{\delta}_k v_k\right) \left(\sum_{l=1}^{3} \boldsymbol{\delta}_l v_l\right)\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i x_i\right) \times \left[\left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial}{\partial x_j}\right) \cdot \left(\rho \sum_{k=1}^{3} \sum_{l=1}^{3} \boldsymbol{\delta}_k \boldsymbol{\delta}_l v_k v_l\right)\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i x_i\right) \times \left[\sum_{j=1}^{3} \sum_{k=1}^{3} \sum_{l=1}^{3} (\boldsymbol{\delta}_j \cdot \boldsymbol{\delta}_k) \boldsymbol{\delta}_l \frac{\partial}{\partial x_j} \rho v_k v_l\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i x_i\right) \times \left(\sum_{j=1}^{3} \sum_{k=1}^{3} \sum_{l=1}^{3} \delta_{jk} \boldsymbol{\delta}_l \frac{\partial}{\partial x_j} \rho v_k v_l\right) \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i x_i\right) \times \left(\sum_{k=1}^{3} \sum_{l=1}^{3} \boldsymbol{\delta}_l \frac{\partial}{\partial x_k} \rho v_k v_l\right) \\
&= \sum_{i=1}^{3} \sum_{k=1}^{3} \sum_{l=1}^{3} (\boldsymbol{\delta}_i \times \boldsymbol{\delta}_l) x_i \frac{\partial}{\partial x_k} \rho v_k v_l \\
&= \sum_{i=1}^{3} \sum_{k=1}^{3} \sum_{l=1}^{3} (\boldsymbol{\delta}_i \times \boldsymbol{\delta}_l) \left(\frac{\partial}{\partial x_k} \rho v_k x_i v_l - \rho v_k v_l \frac{\partial}{\partial x_k} x_i\right) \\
&= \sum_{i=1}^{3} \sum_{k=1}^{3} \sum_{l=1}^{3} \sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{ilm} \left(\frac{\partial}{\partial x_k} \rho v_k x_i v_l - \rho v_k v_l \delta_{ik}\right) \\
&= \sum_{i=1}^{3} \sum_{k=1}^{3} \sum_{l=1}^{3} \sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{ilm} \frac{\partial}{\partial x_k} \rho v_k x_i v_l - \sum_{i=1}^{3} \sum_{k=1}^{3} \sum_{l=1}^{3} \sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{ilm} \rho v_k v_l \delta_{ik} \\
&= \sum_{k=1}^{3} \left(\sum_{i=1}^{3} \sum_{l=1}^{3} \sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{ilm} \frac{\partial}{\partial x_k} \rho v_k x_i v_l\right) - \sum_{k=1}^{3} \sum_{l=1}^{3} \sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{klm} \rho v_k v_l \\
&= \sum_{k=1}^{3} \frac{\partial}{\partial x_k} \rho v_k \left(\sum_{i=1}^{3} \sum_{l=1}^{3} \sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{ilm} x_i v_l\right) - \rho \left(\sum_{k=1}^{3} \sum_{l=1}^{3} \sum_{m=1}^{3} \boldsymbol{\delta}_m \varepsilon_{klm} v_k v_l\right) \\
&= \nabla \cdot \rho \mathbf{v}(\mathbf{r} \times \mathbf{v}) - \underbrace{\rho(\mathbf{v} \times \mathbf{v})}_{=\mathbf{0}} \\
&= \nabla \cdot \rho \mathbf{v}(\mathbf{r} \times \mathbf{v})
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{r} \times (\nabla p) &= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i x_i\right) \times \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial}{\partial x_j} p\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \times \boldsymbol{\delta}_j) x_i \frac{\partial}{\partial x_j} p \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \times \boldsymbol{\delta}_j) \left(\frac{\partial}{\partial x_j} x_i p - p \frac{\partial}{\partial x_j} x_i\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{ijk} \left(\frac{\partial}{\partial x_j} x_i p - p\delta_{ij}\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{ijk} \frac{\partial}{\partial x_j} x_i p - \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{ijk} p \delta_{ij} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \delta_{jl} \boldsymbol{\delta}_k \varepsilon_{ilk} \frac{\partial}{\partial x_j} x_i p - \underbrace{\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{jjk} p}_{=\,\mathbf{0}} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} (\boldsymbol{\delta}_j \cdot \boldsymbol{\delta}_l) \boldsymbol{\delta}_k \varepsilon_{ilk} \frac{\partial}{\partial x_j} x_i p \\
&= \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial}{\partial x_j}\right) \cdot \left(\sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_l \boldsymbol{\delta}_k \varepsilon_{ilk} x_i p\right) \\
&= \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial}{\partial x_j}\right) \cdot \left(\sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_l \boldsymbol{\delta}_k \varepsilon_{ikl} x_i p\right)^{\dagger} \\
&= \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial}{\partial x_j}\right) \cdot \left[\sum_{i=1}^{3}\sum_{k=1}^{3} (\boldsymbol{\delta}_i \times \boldsymbol{\delta}_k) \boldsymbol{\delta}_k x_i p\right]^{\dagger} \\
&= \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial}{\partial x_j}\right) \cdot \left[\left(\sum_{i=1}^{3} \boldsymbol{\delta}_i x_i\right) \times \left(\sum_{k=1}^{3} \boldsymbol{\delta}_k \boldsymbol{\delta}_k p\right)\right]^{\dagger} \\
&= \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial}{\partial x_j}\right) \cdot \left[\left(\sum_{i=1}^{3} \boldsymbol{\delta}_i x_i\right) \times p \left(\sum_{k=1}^{3}\sum_{m=1}^{3} \delta_{km} \boldsymbol{\delta}_k \boldsymbol{\delta}_m\right)\right]^{\dagger} \\
&= \nabla \cdot (\mathbf{r} \times p\boldsymbol{\delta})^{\dagger}
\end{aligned}
$$

$$\begin{aligned}
\mathbf{r}\times(\nabla\cdot\boldsymbol{\tau}) &= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i x_i\right)\times\left[\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}\right)\cdot\left(\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_k\boldsymbol{\delta}_l\tau_{kl}\right)\right]\\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i x_i\right)\times\left[\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_j\cdot\boldsymbol{\delta}_k)\boldsymbol{\delta}_l\frac{\partial}{\partial x_j}\tau_{kl}\right]\\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i x_i\right)\times\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\delta_{jk}\boldsymbol{\delta}_l\frac{\partial}{\partial x_j}\tau_{kl}\right)\\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i x_i\right)\times\left(\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_l\frac{\partial}{\partial x_k}\tau_{kl}\right)\\
&= \sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_i\times\boldsymbol{\delta}_l)x_i\frac{\partial}{\partial x_k}\tau_{kl}\\
&= \sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_i\times\boldsymbol{\delta}_l)\left(\frac{\partial}{\partial x_k}x_i\tau_{kl}-\tau_{kl}\frac{\partial}{\partial x_k}x_i\right)\\
&= \sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\varepsilon_{ilm}\left(\frac{\partial}{\partial x_k}x_i\tau_{kl}-\tau_{kl}\delta_{ik}\right)\\
&= \sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\varepsilon_{ilm}\frac{\partial}{\partial x_k}x_i\tau_{kl}-\sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\varepsilon_{ilm}\tau_{kl}\delta_{ik}\\
&= \sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}\delta_{nk}\boldsymbol{\delta}_m\varepsilon_{ilm}\frac{\partial}{\partial x_n}x_i\tau_{kl}-\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\varepsilon_{klm}\tau_{kl}\\
&= \sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}(\boldsymbol{\delta}_n\cdot\boldsymbol{\delta}_k)\boldsymbol{\delta}_m\varepsilon_{ilm}\frac{\partial}{\partial x_n}x_i\tau_{kl}-\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m(-\varepsilon_{mlk})\tau_{kl}\\
&= \left(\sum_{n=1}^{3}\boldsymbol{\delta}_n\frac{\partial}{\partial x_n}\right)\cdot\left(\sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_k\boldsymbol{\delta}_m\varepsilon_{ilm}x_i\tau_{kl}\right)+\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\varepsilon_{mlk}\tau_{kl}\\
&= \left(\sum_{n=1}^{3}\boldsymbol{\delta}_n\frac{\partial}{\partial x_n}\right)\cdot\left(\sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_k\boldsymbol{\delta}_m\varepsilon_{ilk}x_i\tau_{ml}\right)^{\dagger}+\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}\sum_{p=1}^{3}\boldsymbol{\delta}_m\delta_{lp}\delta_{kn}\varepsilon_{mlk}\tau_{np}\\
&= \left(\sum_{n=1}^{3}\boldsymbol{\delta}_n\frac{\partial}{\partial x_n}\right)\cdot\left[\sum_{i=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}(\boldsymbol{\delta}_i\times\boldsymbol{\delta}_l)\boldsymbol{\delta}_m x_i\tau_{ml}\right]^{\dagger}+\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}\sum_{p=1}^{3}\boldsymbol{\delta}_m(\boldsymbol{\delta}_l\cdot\boldsymbol{\delta}_p)(\boldsymbol{\delta}_k\cdot\boldsymbol{\delta}_n)\varepsilon_{mlk}\tau_{np}\\
&= \left(\sum_{n=1}^{3}\boldsymbol{\delta}_n\frac{\partial}{\partial x_n}\right)\cdot\left[\left(\sum_{i=1}^{3}\boldsymbol{\delta}_i x_i\right)\times\left(\sum_{l=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_l\boldsymbol{\delta}_m\tau_{ml}\right)\right]^{\dagger}+\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}\sum_{p=1}^{3}\boldsymbol{\delta}_m(\boldsymbol{\delta}_l\boldsymbol{\delta}_k:\boldsymbol{\delta}_n\boldsymbol{\delta}_p)\varepsilon_{mlk}\tau_{np}\\
&= \left(\sum_{n=1}^{3}\boldsymbol{\delta}_n\frac{\partial}{\partial x_n}\right)\cdot\left[\left(\sum_{i=1}^{3}\boldsymbol{\delta}_i x_i\right)\times\left(\sum_{l=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_l\boldsymbol{\delta}_m\tau_{lm}\right)^{\dagger}\right]^{\dagger}+\left(\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\boldsymbol{\delta}_l\boldsymbol{\delta}_k\varepsilon_{mlk}\right):\left(\sum_{n=1}^{3}\sum_{p=1}^{3}\boldsymbol{\delta}_n\boldsymbol{\delta}_p\tau_{np}\right)\\
&= \nabla\cdot(\mathbf{r}\times\boldsymbol{\tau}^{\dagger})^{\dagger}+\boldsymbol{\varepsilon}:\boldsymbol{\tau}
\end{aligned}$$

Therefore, the equation of change for angular momentum is found.

$$\boxed{\frac{\partial}{\partial t}\rho(\mathbf{r}\times\mathbf{v}) = -\nabla\cdot\rho\mathbf{v}(\mathbf{r}\times\mathbf{v}) - \nabla\cdot(\mathbf{r}\times p\boldsymbol{\delta})^\dagger - \nabla\cdot(\mathbf{r}\times\boldsymbol{\tau}^\dagger)^\dagger + \mathbf{r}\times\rho\mathbf{g} - \boldsymbol{\varepsilon}:\boldsymbol{\tau}} \tag{3.4-1}$$

Assume that the fluid in the Couette viscometer is Newtonian. Then $\boldsymbol{\tau}$ obeys Newton's generalized law of viscosity and is symmetric ($\tau_{ij} = \tau_{ji}$). As a result, the last term in this equation vanishes.

$$\begin{aligned}
\boldsymbol{\varepsilon}:\boldsymbol{\tau} &= \left(\sum_{i=1}^3\sum_{j=1}^3\sum_{k=1}^3 \boldsymbol{\delta}_i\boldsymbol{\delta}_j\boldsymbol{\delta}_k\varepsilon_{ijk}\right) : \left(\sum_{l=1}^3\sum_{m=1}^3 \boldsymbol{\delta}_l\boldsymbol{\delta}_m\tau_{lm}\right) \\
&= \sum_{i=1}^3\sum_{j=1}^3\sum_{k=1}^3\sum_{l=1}^3\sum_{m=1}^3 \boldsymbol{\delta}_i(\boldsymbol{\delta}_j\boldsymbol{\delta}_k : \boldsymbol{\delta}_l\boldsymbol{\delta}_m)\varepsilon_{ijk}\tau_{lm} \\
&= \sum_{i=1}^3\sum_{j=1}^3\sum_{k=1}^3\sum_{l=1}^3\sum_{m=1}^3 \boldsymbol{\delta}_i(\boldsymbol{\delta}_j\cdot\boldsymbol{\delta}_m)(\boldsymbol{\delta}_k\cdot\boldsymbol{\delta}_l)\varepsilon_{ijk}\tau_{lm} \\
&= \sum_{i=1}^3\sum_{j=1}^3\sum_{k=1}^3\sum_{l=1}^3\sum_{m=1}^3 \boldsymbol{\delta}_i\delta_{jm}\delta_{kl}\varepsilon_{ijk}\tau_{lm} \\
&= \sum_{i=1}^3\sum_{j=1}^3\sum_{k=1}^3 \boldsymbol{\delta}_i\varepsilon_{ijk}\tau_{kj} \\
&= \sum_{i=1}^3\sum_{k=1}^3\sum_{j=1}^3 \boldsymbol{\delta}_i\varepsilon_{ikj}\tau_{jk} \qquad \text{(let } j \text{ be } k \text{ and let } k \text{ be } j\text{)} \\
&= \sum_{i=1}^3\sum_{j=1}^3\sum_{k=1}^3 \boldsymbol{\delta}_i\varepsilon_{ikj}\tau_{jk} \\
&= \sum_{i=1}^3\sum_{j=1}^3\sum_{k=1}^3 \boldsymbol{\delta}_i(-\varepsilon_{ijk})\tau_{jk} \\
&= -\sum_{i=1}^3\sum_{j=1}^3\sum_{k=1}^3 \boldsymbol{\delta}_i\varepsilon_{ijk}\tau_{kj} \\
&= \mathbf{0}
\end{aligned}$$

It's natural to use a cylindrical coordinate system for this viscometer, so the position vector is $\mathbf{r} = r\hat{\mathbf{r}} + z\hat{\mathbf{z}}$. In addition, gravity points down in the negative z-direction: $\mathbf{g} = -g\hat{\mathbf{z}}$.

$$\mathbf{r}\times\rho\mathbf{g} = \begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\theta}} & \hat{\mathbf{z}} \\ r & 0 & z \\ 0 & 0 & -\rho g \end{vmatrix} = \rho g r\hat{\boldsymbol{\theta}}$$

The fluid velocity is assumed to only have an angular component which varies with respect to radius: $\mathbf{v} = v_\theta(r)\hat{\boldsymbol{\theta}}$. Evaluate the quantity $\mathbf{r}\times\mathbf{v}$.

$$\mathbf{r}\times\mathbf{v} = \begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\theta}} & \hat{\mathbf{z}} \\ r & 0 & z \\ 0 & v_\theta(r) & 0 \end{vmatrix} = -zv_\theta(r)\hat{\mathbf{r}} + rv_\theta(r)\hat{\mathbf{z}}$$

Neither of the components of $\mathbf{r} \times \mathbf{v}$ depends on t, so the time derivative vanishes with the additional assumption that the density ρ is constant.

$$\frac{\partial}{\partial t}\rho(\mathbf{r} \times \mathbf{v}) = \mathbf{0}$$

Form the dyadic product of $\mathbf{v}$ and $\mathbf{r} \times \mathbf{v}$

$$\begin{aligned}\mathbf{v}(\mathbf{r} \times \mathbf{v}) &= v_\theta(r)\boldsymbol{\delta}_\theta[-zv_\theta(r)\boldsymbol{\delta}_r + rv_\theta(r)\boldsymbol{\delta}_z] \\ &= -z[v_\theta(r)]^2\boldsymbol{\delta}_\theta\boldsymbol{\delta}_r + r[v_\theta(r)]^2\boldsymbol{\delta}_\theta\boldsymbol{\delta}_z\end{aligned}$$

and then use formulas (J), (K), and (L) on page 834 to find the divergence of this second-order tensor in cylindrical coordinates.

$$\begin{aligned}\nabla \cdot \rho\mathbf{v}(\mathbf{r} \times \mathbf{v}) &= \rho\nabla \cdot \mathbf{v}(\mathbf{r} \times \mathbf{v}) \\ &= \rho\Bigg[\underbrace{\frac{1}{r}\frac{\partial}{\partial\theta}\{-z[v_\theta(r)]^2\}\hat{\mathbf{r}}}_{=\mathbf{0}} + \frac{1}{r}\{-z[v_\theta(r)]^2\}\hat{\boldsymbol{\theta}} + \underbrace{\frac{1}{r}\frac{\partial}{\partial\theta}\{r[v_\theta(r)]^2\}\hat{\mathbf{z}}}_{=\mathbf{0}}\Bigg] \\ &= -\frac{\rho z}{r}[v_\theta(r)]^2\hat{\boldsymbol{\theta}}\end{aligned}$$

Because of the centrifugal and gravitational forces on the fluid, the pressure is assumed to vary in r and z: $p = p(r, z)$. Evaluate the quantity $(\mathbf{r} \times p\boldsymbol{\delta})^\dagger$

$$\begin{aligned}(\mathbf{r} \times p\boldsymbol{\delta})^\dagger &= [(r\boldsymbol{\delta}_r + z\boldsymbol{\delta}_z) \times (p\boldsymbol{\delta}_r\boldsymbol{\delta}_r + p\boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta + p\boldsymbol{\delta}_z\boldsymbol{\delta}_z)]^\dagger \\ &= [rp(\boldsymbol{\delta}_r \times \boldsymbol{\delta}_r)\boldsymbol{\delta}_r + rp(\boldsymbol{\delta}_r \times \boldsymbol{\delta}_\theta)\boldsymbol{\delta}_\theta + rp(\boldsymbol{\delta}_r \times \boldsymbol{\delta}_z)\boldsymbol{\delta}_z \\ &\qquad + zp(\boldsymbol{\delta}_z \times \boldsymbol{\delta}_r)\boldsymbol{\delta}_r + zp(\boldsymbol{\delta}_z \times \boldsymbol{\delta}_\theta)\boldsymbol{\delta}_\theta + zp(\boldsymbol{\delta}_z \times \boldsymbol{\delta}_z)\boldsymbol{\delta}_z]^\dagger \\ &= [rp(\mathbf{0})\boldsymbol{\delta}_r + rp(\boldsymbol{\delta}_z)\boldsymbol{\delta}_\theta + rp(-\boldsymbol{\delta}_\theta)\boldsymbol{\delta}_z \\ &\qquad + zp(\boldsymbol{\delta}_\theta)\boldsymbol{\delta}_r + zp(-\boldsymbol{\delta}_r)\boldsymbol{\delta}_\theta + zp(\mathbf{0})\boldsymbol{\delta}_z]^\dagger \\ &= (rp\boldsymbol{\delta}_z\boldsymbol{\delta}_\theta - rp\boldsymbol{\delta}_\theta\boldsymbol{\delta}_z + zp\boldsymbol{\delta}_\theta\boldsymbol{\delta}_r - zp\boldsymbol{\delta}_r\boldsymbol{\delta}_\theta)^\dagger \\ &= -rp\boldsymbol{\delta}_z\boldsymbol{\delta}_\theta + rp\boldsymbol{\delta}_\theta\boldsymbol{\delta}_z - zp\boldsymbol{\delta}_\theta\boldsymbol{\delta}_r + zp\boldsymbol{\delta}_r\boldsymbol{\delta}_\theta\end{aligned}$$

and then use formulas (J), (K), and (L) on page 834 once more to find its divergence.

$$\begin{aligned}\nabla \cdot (\mathbf{r} \times p\boldsymbol{\delta})^\dagger &= \underbrace{\frac{1}{r}\frac{\partial}{\partial\theta}(-zp)\hat{\mathbf{r}}}_{=\mathbf{0}} + \frac{1}{r^2}\frac{\partial}{\partial r}[r^2(zp)]\hat{\boldsymbol{\theta}} + \frac{\partial}{\partial z}(-rp)\hat{\boldsymbol{\theta}} + \frac{-zp - zp}{r}\hat{\boldsymbol{\theta}} + \underbrace{\frac{1}{r}\frac{\partial}{\partial\theta}(rp)\hat{\boldsymbol{z}}}_{=\mathbf{0}} \\ &= \left[\frac{z}{r^2}\frac{\partial}{\partial r}(r^2p) - r\frac{\partial p}{\partial z} - \frac{2z}{r}p\right]\hat{\boldsymbol{\theta}}\end{aligned}$$

In order to simplify the final term, start by evaluating $(\mathbf{r} \times \boldsymbol{\tau}^\dagger)^\dagger$. Looking at page 844, only $\tau_{r\theta}$ and $\tau_{\theta r}$ are relevant with the assumed velocity.

$$\begin{aligned}(\mathbf{r} \times \boldsymbol{\tau}^\dagger)^\dagger &= [(r\boldsymbol{\delta}_r + z\boldsymbol{\delta}_z) \times (\tau_{r\theta}\boldsymbol{\delta}_r\boldsymbol{\delta}_\theta + \tau_{\theta r}\boldsymbol{\delta}_\theta\boldsymbol{\delta}_r)^\dagger]^\dagger \\ &= [(r\boldsymbol{\delta}_r + z\boldsymbol{\delta}_z) \times (\tau_{\theta r}\boldsymbol{\delta}_r\boldsymbol{\delta}_\theta + \tau_{r\theta}\boldsymbol{\delta}_\theta\boldsymbol{\delta}_r)]^\dagger \\ &= [r\tau_{\theta r}(\boldsymbol{\delta}_r \times \boldsymbol{\delta}_r)\boldsymbol{\delta}_\theta + r\tau_{r\theta}(\boldsymbol{\delta}_r \times \boldsymbol{\delta}_\theta)\boldsymbol{\delta}_r + z\tau_{\theta r}(\boldsymbol{\delta}_z \times \boldsymbol{\delta}_r)\boldsymbol{\delta}_\theta + z\tau_{r\theta}(\boldsymbol{\delta}_z \times \boldsymbol{\delta}_\theta)\boldsymbol{\delta}_r]^\dagger \\ &= [r\tau_{\theta r}(\mathbf{0})\boldsymbol{\delta}_\theta + r\tau_{r\theta}(\boldsymbol{\delta}_z)\boldsymbol{\delta}_r + z\tau_{\theta r}(\boldsymbol{\delta}_\theta)\boldsymbol{\delta}_\theta + z\tau_{r\theta}(-\boldsymbol{\delta}_r)\boldsymbol{\delta}_r]^\dagger \\ &= (r\tau_{r\theta}\boldsymbol{\delta}_z\boldsymbol{\delta}_r + z\tau_{\theta r}\boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta - z\tau_{r\theta}\boldsymbol{\delta}_r\boldsymbol{\delta}_r)^\dagger \\ &= r\tau_{r\theta}\boldsymbol{\delta}_r\boldsymbol{\delta}_z + z\tau_{\theta r}\boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta - z\tau_{r\theta}\boldsymbol{\delta}_r\boldsymbol{\delta}_r\end{aligned}$$

Use formulas (J), (K), and (L) on page 834 once more to find its divergence.

$$\begin{aligned}\nabla\cdot(\mathbf{r}\times\boldsymbol{\tau}^\dagger)^\dagger &= \frac{1}{r}\frac{\partial}{\partial r}[r(-z\tau_{r\theta})]\hat{\mathbf{r}} - \frac{z\tau_{\theta r}}{r}\hat{\mathbf{r}} + \underbrace{\frac{1}{r}\frac{\partial}{\partial\theta}(z\tau_{\theta r})\hat{\boldsymbol{\theta}}}_{=\mathbf{0}} + \frac{1}{r}\frac{\partial}{\partial r}[r(r\tau_{r\theta})]\hat{\mathbf{z}} \\ &= -\frac{z}{r}\left[\frac{d}{dr}(r\tau_{r\theta}) + \tau_{\theta r}\right]\hat{\mathbf{r}} + \frac{1}{r}\frac{d}{dr}(r^2\tau_{r\theta})\hat{\mathbf{z}}\end{aligned}$$

With these simplifications, the equation of change for angular momentum becomes

$$\mathbf{0} = \frac{\rho z}{r}[v_\theta(r)]^2\hat{\boldsymbol{\theta}} - \left[\frac{z}{r^2}\frac{\partial}{\partial r}(r^2 p) - r\frac{\partial p}{\partial z} - \frac{2z}{r}p\right]\hat{\boldsymbol{\theta}} - \left\{-\frac{z}{r}\left[\frac{d}{dr}(r\tau_{r\theta}) + \tau_{\theta r}\right]\hat{\mathbf{r}} + \frac{1}{r}\frac{d}{dr}(r^2\tau_{r\theta})\hat{\mathbf{z}}\right\} + \rho g r\hat{\boldsymbol{\theta}}.$$

Dot both sides by $\hat{\mathbf{z}}$.

$$0 = -\frac{1}{r}\frac{d}{dr}(r^2\tau_{r\theta})$$

Multiply both sides by $-r$.

$$\frac{d}{dr}(r^2\tau_{r\theta}) = 0 \qquad (3C.4\text{-}1)$$

Alternatively, dot both sides by $\hat{\mathbf{r}}$.

$$0 = \frac{z}{r}\left[\frac{d}{dr}(r\tau_{r\theta}) + \tau_{\theta r}\right]$$

Multiply both sides by r and divide both sides by z.

$$\frac{d}{dr}(r\tau_{r\theta}) + \tau_{\theta r} = 0$$

$\boldsymbol{\tau}$ is symmetric, so $\tau_{r\theta} = \tau_{\theta r}$.

$$\frac{d}{dr}(r\tau_{r\theta}) + \tau_{r\theta} = 0$$

Expand the derivative using the product rule.

$$\tau_{r\theta} + r\frac{d\tau_{r\theta}}{dr} + \tau_{r\theta} = 0$$

$$r\frac{d\tau_{r\theta}}{dr} + 2\tau_{r\theta} = 0$$

Multiply both sides by r.

$$r^2\frac{d\tau_{r\theta}}{dr} + 2r\tau_{r\theta} = 0$$

Therefore, by the product rule,

$$\frac{d}{dr}(r^2\tau_{r\theta}) = 0. \qquad (3C.4\text{-}1)$$