Problem 3C.5

Two-phase interfacial boundary conditions. In §2.1, boundary conditions for solving viscous flow problems were given. At that point no mention was made of the role of interfacial tension. At the interface between two immiscible fluids, I and II, the following boundary condition should be used:[7]

$$\mathbf{n}^{\mathrm{I}}(p^{\mathrm{I}} - p^{\mathrm{II}}) + [\mathbf{n}^{\mathrm{I}} \cdot (\boldsymbol{\tau}^{\mathrm{I}} - \boldsymbol{\tau}^{\mathrm{II}})] = \mathbf{n}^{\mathrm{I}} \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\sigma \tag{3C.5-1}$$

This is essentially a momentum balance written for an interfacial element dS with no matter passing through it, and with no interfacial mass or viscosity. Here $\mathbf{n}^{\mathrm{I}}$ is the unit vector normal to dS and pointing into phase I. The quantities R_1 and R_2 are the principal radii of curvature at dS, and each of these is positive if its center lies in phase I. The sum $(1/R_1) + (1/R_2)$ can also be expressed as $(\nabla \cdot \mathbf{n}^{\mathrm{I}})$. The quantity σ is the interfacial tension, assumed constant.

(a) Show that, for a spherical droplet of I at rest in a second medium II, *Laplace's equation*

$$p^{\mathrm{I}} - p^{\mathrm{II}} = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\sigma \tag{3C.5-2}$$

relates the pressures inside and outside the droplet. Is the pressure in phase I greater than that in phase II, or the reverse? What is the relation between the pressures at a planar interface?

(b) Show that Eq. 3C.5-1 leads to the following dimensionless boundary condition.

$$\begin{aligned}\mathbf{n}^{\mathrm{I}}(\breve{\mathscr{P}}^{\mathrm{I}} - \breve{\mathscr{P}}^{\mathrm{II}}) + \mathbf{n}^{\mathrm{I}}\left[\!\left[\frac{\rho^{\mathrm{II}} - \rho^{\mathrm{I}}}{\rho^{\mathrm{I}}}\right]\!\right]\left[\!\left[\frac{gl_0}{v_0^2}\right]\!\right]\breve{h} \\ - \left[\!\left[\frac{\mu^{\mathrm{I}}}{l_0 v_0 \rho^{\mathrm{I}}}\right]\!\right][\mathbf{n}^{\mathrm{I}} \cdot \breve{\dot{\boldsymbol{\gamma}}}^{\mathrm{I}}] + \left[\!\left[\frac{\mu^{\mathrm{II}}}{l_0 v_0 \rho^{\mathrm{II}}}\right]\!\right][\mathbf{n}^{\mathrm{I}} \cdot \breve{\dot{\boldsymbol{\gamma}}}^{\mathrm{II}}]\left[\!\left[\frac{\rho^{\mathrm{II}}}{\rho^{\mathrm{I}}}\right]\!\right] = \mathbf{n}^{\mathrm{I}}\left(\frac{1}{\breve{R}_1} + \frac{1}{\breve{R}_2}\right)\left[\!\left[\frac{\sigma}{l_0 v_0^2 \rho^{\mathrm{I}}}\right]\!\right]\end{aligned} \tag{3C.5-3}$$

in which $\breve{h} = (h - h_0)/l_0$ is the dimensionless elevation of dS, $\breve{\dot{\boldsymbol{\gamma}}}^{\mathrm{I}}$ and $\breve{\dot{\boldsymbol{\gamma}}}^{\mathrm{II}}$ are dimensionless rate-of-deformation tensors, and $\breve{R}_1 = R_1/l_0$ and $\breve{R}_2 = R_2/l_0$ are dimensionless radii of curvature. Furthermore

$$\breve{\mathscr{P}}^{\mathrm{I}} = \frac{p^{\mathrm{I}} - p_0 + \rho^{\mathrm{I}} g(h - h_0)}{\rho^{\mathrm{I}} v_0^2}$$

$$\breve{\mathscr{P}}^{\mathrm{II}} = \frac{p^{\mathrm{II}} - p_0 + \rho^{\mathrm{II}} g(h - h_0)}{\rho^{\mathrm{I}} v_0^2} \tag{3C.5-4, 5}$$

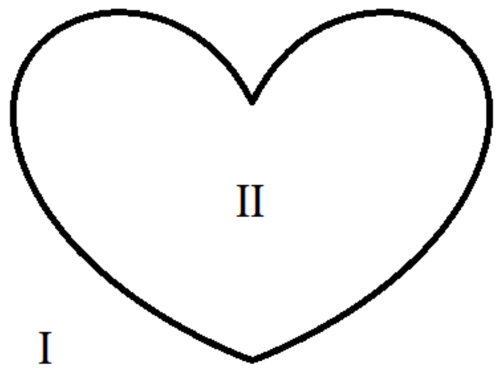
In the above, the zero-subscripted quantities are the scale factors, valid in both phases. Identify the dimensionless groups that appear in Eq. 3C.5-3.

(c) Show how the result in (b) simplifies to Eq. 3.7-36 under the assumptions made in Example 3.7-2.

Solution

[7] L. Landau and E. M. Lifshitz, *Fluid Mechanics*, Pergamon, Oxford, 2nd edition (1987), Eq. 61.13. More general formulas including the excess density and viscosity have been developed by L. E. Scriven, *Chem. Eng. Sci.*, **12**, 98–108 (1960).

The boundary condition will be derived here by using an integral formulation. Consider two immiscible fluids in space that have an arbitrary two-dimensional boundary. In order for the outward normal unit vector of this boundary to be coincident with $\mathbf{n}^{\mathrm{I}}$, the normal unit vector pointing into phase I, let the fluid inside the boundary be phase II and let the fluid outside be phase I.



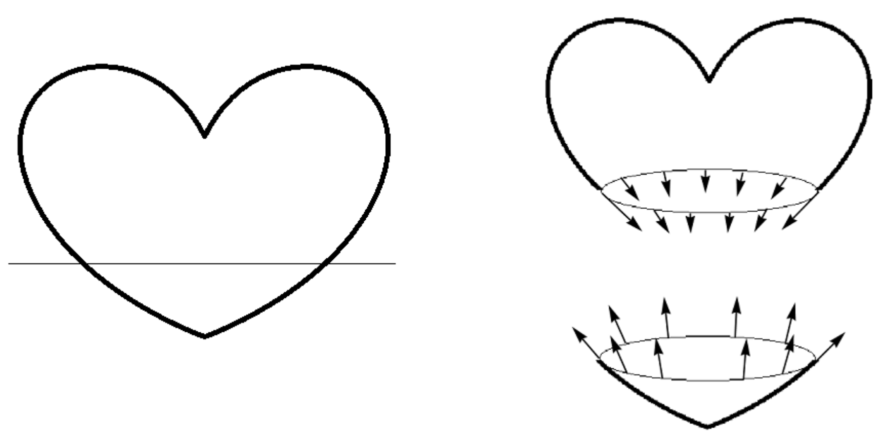
Apply Newton's second law to the boundary.

$$\sum \mathbf{F} = m\mathbf{a}$$

Since there is no interfacial mass, the right side is zero.

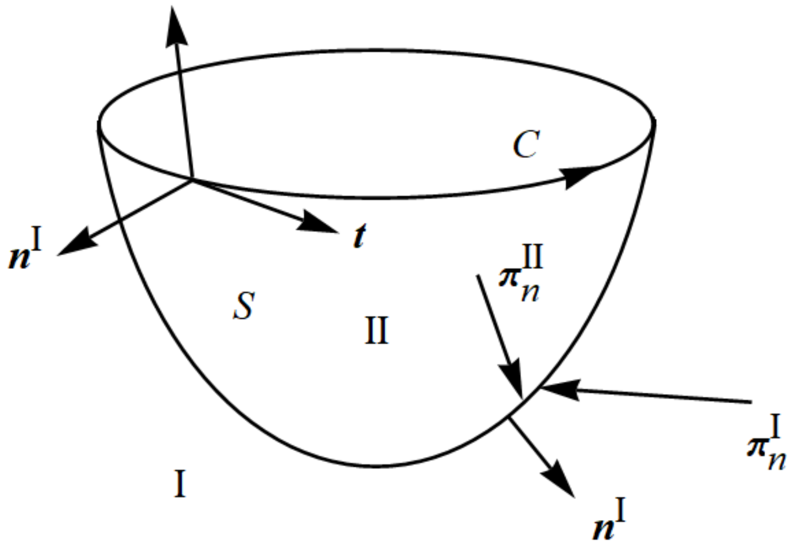
$$\sum \mathbf{F} = \mathbf{0}$$

The collection of forces (per unit area) acting on the boundary from one phase can be obtained from the second-order stress tensor $\boldsymbol{\pi} = p\boldsymbol{\delta} + \boldsymbol{\tau}$. From it, the forces resulting from the pressure, the normal stress due to viscosity, and the shearing stress due to viscosity will be accounted for. Since there are two phases, there will be two respective tensors, $\boldsymbol{\pi}^{\mathrm{I}}$ and $\boldsymbol{\pi}^{\mathrm{II}}$, and the total force is obtained by integrating them over the boundary S. In addition to these forces, the surface tension σ in the boundary also needs to be considered. It's assumed that σ is constant and that the tension is tangential to the boundary.



This figure illustrates the forces due to surface tension in one cross-section of the boundary. As σ has units of force per unit length, the total force due to surface tension is obtained by integrating σ over the bounding curve C.

Zoom in on the severed piece and draw a free-body diagram.



In this figure, $\mathbf{t}$ is a unit vector tangential to the bounding curve C at every point. The direction of the force due to surface tension is then $\mathbf{n}^{\mathrm{I}} \times \mathbf{t}$. The subscript n is included on the stress tensors to indicate that the forces (per unit area) are acting on a surface whose normal unit vector is $\mathbf{n}^{\mathrm{I}}$. Newton's second law becomes

$$\oint_C \sigma(\mathbf{n}^{\mathrm{I}} \times \mathbf{t})\, dt + \iint_S \boldsymbol{\pi}^{\mathrm{I}} \cdot d\mathbf{S} + \iint_S \boldsymbol{\pi}^{\mathrm{II}} \cdot d\mathbf{S} = \mathbf{0}.$$

Because $\boldsymbol{\pi}^{\mathrm{I}}$ points towards the surface and $\mathbf{n}^{\mathrm{I}}$ points away from it, the dot product results in a minus sign. Since $\boldsymbol{\pi}^{\mathrm{II}}$ and $\mathbf{n}^{\mathrm{I}}$ both point away from the surface, this dot product is positive.

$$\begin{aligned}
\mathbf{0} &= \oint_C \sigma(\mathbf{n}^{\mathrm{I}} \times \mathbf{t})\, dt + \iint_S \boldsymbol{\pi}^{\mathrm{I}} \cdot (-\mathbf{n}^{\mathrm{I}})\, dS + \iint_S \boldsymbol{\pi}^{\mathrm{II}} \cdot \mathbf{n}^{\mathrm{I}}\, dS \\
&= \oint_C \sigma(\mathbf{n}^{\mathrm{I}} \times \mathbf{t})\, dt - \iint_S \boldsymbol{\pi}^{\mathrm{I}} \cdot \mathbf{n}^{\mathrm{I}}\, dS + \iint_S \boldsymbol{\pi}^{\mathrm{II}} \cdot \mathbf{n}^{\mathrm{I}}\, dS \\
&= \oint_C \sigma(\mathbf{n}^{\mathrm{I}} \times \mathbf{t})\, dt - \iint_S (p^{\mathrm{I}}\boldsymbol{\delta} + \boldsymbol{\tau}^{\mathrm{I}}) \cdot \mathbf{n}^{\mathrm{I}}\, dS + \iint_S (p^{\mathrm{II}}\boldsymbol{\delta} + \boldsymbol{\tau}^{\mathrm{II}}) \cdot \mathbf{n}^{\mathrm{I}}\, dS \\
&= \oint_C \sigma(\mathbf{n}^{\mathrm{I}} \times \mathbf{t})\, dt - \iint_S [p^{\mathrm{I}}(\boldsymbol{\delta} \cdot \mathbf{n}^{\mathrm{I}}) + \boldsymbol{\tau}^{\mathrm{I}} \cdot \mathbf{n}^{\mathrm{I}}]\, dS + \iint_S [p^{\mathrm{II}}(\boldsymbol{\delta} \cdot \mathbf{n}^{\mathrm{I}}) + \boldsymbol{\tau}^{\mathrm{II}} \cdot \mathbf{n}^{\mathrm{I}}]\, dS
\end{aligned}$$

Note that

$$\begin{aligned}
\boldsymbol{\delta} \cdot \mathbf{n}^{\mathrm{I}} &= \left(\sum_{i=1}^{3}\sum_{j=1}^{3} \boldsymbol{\delta}_i \boldsymbol{\delta}_j \delta_{ij}\right) \cdot \left(\sum_{k=1}^{3} \boldsymbol{\delta}_k n_k^{\mathrm{I}}\right) \\
&= \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \boldsymbol{\delta}_j\right) \cdot \left(\sum_{k=1}^{3} \boldsymbol{\delta}_k n_k^{\mathrm{I}}\right) \\
&= \sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_j (\boldsymbol{\delta}_j \cdot \boldsymbol{\delta}_k) n_k^{\mathrm{I}} = \sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_j \delta_{jk} n_k^{\mathrm{I}} = \sum_{k=1}^{3} \boldsymbol{\delta}_k n_k^{\mathrm{I}} = \mathbf{n}^{\mathrm{I}}.
\end{aligned}$$

As a result,

$$\begin{aligned}
\mathbf{0} &= \oint_C \sigma(\mathbf{n}^{\text{I}} \times \mathbf{t})\, dt - \iint_S (p^{\text{I}}\mathbf{n}^{\text{I}} + \boldsymbol{\tau}^{\text{I}} \cdot \mathbf{n}^{\text{I}})\, dS + \iint_S (p^{\text{II}}\mathbf{n}^{\text{I}} + \boldsymbol{\tau}^{\text{II}} \cdot \mathbf{n}^{\text{I}})\, dS \\
&= \oint_C \sigma(\mathbf{n}^{\text{I}} \times \mathbf{t})\, dt - \iint_S (\mathbf{n}^{\text{I}}p^{\text{I}} + \mathbf{n}^{\text{I}} \cdot \boldsymbol{\tau}^{\text{I}})\, dS + \iint_S (\mathbf{n}^{\text{I}}p^{\text{II}} + \mathbf{n}^{\text{I}} \cdot \boldsymbol{\tau}^{\text{II}})\, dS \\
&= \oint_C \sigma(\mathbf{n}^{\text{I}} \times \mathbf{t})\, dt - \iint_S (\mathbf{n}^{\text{I}}p^{\text{I}} + \mathbf{n}^{\text{I}} \cdot \boldsymbol{\tau}^{\text{I}} - \mathbf{n}^{\text{I}}p^{\text{II}} - \mathbf{n}^{\text{I}} \cdot \boldsymbol{\tau}^{\text{II}})\, dS \\
&= \oint_C \sigma(\mathbf{n}^{\text{I}} \times \mathbf{t})\, dt - \iint_S [\mathbf{n}^{\text{I}}(p^{\text{I}} - p^{\text{II}}) + \mathbf{n}^{\text{I}} \cdot (\boldsymbol{\tau}^{\text{I}} - \boldsymbol{\tau}^{\text{II}})]\, dS.
\end{aligned}$$

Bring the surface integral to the left side.

$$\iint_S [\mathbf{n}^{\text{I}}(p^{\text{I}} - p^{\text{II}}) + \mathbf{n}^{\text{I}} \cdot (\boldsymbol{\tau}^{\text{I}} - \boldsymbol{\tau}^{\text{II}})]\, dS = \oint_C \sigma(\mathbf{n}^{\text{I}} \times \mathbf{t})\, dt$$

Since σ is constant, it can be brought in front of the line integral.

$$\iint_S [\mathbf{n}^{\text{I}}(p^{\text{I}} - p^{\text{II}}) + \mathbf{n}^{\text{I}} \cdot (\boldsymbol{\tau}^{\text{I}} - \boldsymbol{\tau}^{\text{II}})]\, dS = \sigma \oint_C (\mathbf{n}^{\text{I}} \times \mathbf{t})\, dt \tag{1}$$

All that's left to do is to use Stokes's theorem to turn this closed loop integral on the right side into a surface integral. The issue is that the integrand has a cross product rather than a dot product, so the theorem can't be applied now. Consider the dot product of an arbitrary vector $\mathbf{A}$ with the closed loop integral and then use the triple product vector identity,
$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$.

$$\begin{aligned}
\mathbf{A} \cdot \oint_C (\mathbf{n}^{\text{I}} \times \mathbf{t})\, dt &= \oint_C \mathbf{A} \cdot (\mathbf{n}^{\text{I}} \times \mathbf{t})\, dt \\
&= \oint_C \mathbf{t} \cdot (\mathbf{A} \times \mathbf{n}^{\text{I}})\, dt \\
&= \oint_C (\mathbf{A} \times \mathbf{n}^{\text{I}}) \cdot \mathbf{t}\, dt \\
&= \oint_C (\mathbf{A} \times \mathbf{n}^{\text{I}}) \cdot d\mathbf{t} \qquad \text{(now Stokes's law can be applied)} \\
&= \iint_S \nabla \times (\mathbf{A} \times \mathbf{n}^{\text{I}}) \cdot d\mathbf{S} \\
&= \iint_S \left(\sum_{i=1}^3 \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left[\left(\sum_{j=1}^3 \boldsymbol{\delta}_j A_j\right) \times \left(\sum_{k=1}^3 \boldsymbol{\delta}_k n_k^{\text{I}}\right)\right] \cdot d\mathbf{S} \\
&= \iint_S \left(\sum_{i=1}^3 \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left[\sum_{j=1}^3 \sum_{k=1}^3 (\boldsymbol{\delta}_j \times \boldsymbol{\delta}_k) A_j n_k^{\text{I}}\right] \cdot d\mathbf{S} \\
&= \iint_S \left(\sum_{i=1}^3 \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left(\sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 \boldsymbol{\delta}_l \varepsilon_{jkl} A_j n_k^{\text{I}}\right) \cdot d\mathbf{S} \\
&= \iint_S \left[\sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 (\boldsymbol{\delta}_i \times \boldsymbol{\delta}_l) \varepsilon_{jkl} \frac{\partial}{\partial x_i} A_j n_k^{\text{I}}\right] \cdot d\mathbf{S}
\end{aligned}$$

Because the components of **A** are constant, they can be brought in front of the derivative.

$$
\begin{aligned}
\mathbf{A}\cdot\oint_C(\mathbf{n}^{\mathrm{I}}\times\mathbf{t})\,dt &= \iint_S\left[\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_i\times\boldsymbol{\delta}_l)\varepsilon_{jkl}A_j\frac{\partial n_k^{\mathrm{I}}}{\partial x_i}\right]\cdot d\mathbf{S} \\
&= \iint_S\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\varepsilon_{ilm}\varepsilon_{jkl}A_j\frac{\partial n_k^{\mathrm{I}}}{\partial x_i}\right)\cdot d\mathbf{S} \\
&= \iint_S\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\varepsilon_{mil}\varepsilon_{jkl}A_j\frac{\partial n_k^{\mathrm{I}}}{\partial x_i}\right)\cdot d\mathbf{S} \\
&= \iint_S\left[\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m(\delta_{mj}\delta_{ik}-\delta_{mk}\delta_{ij})A_j\frac{\partial n_k^{\mathrm{I}}}{\partial x_i}\right]\cdot d\mathbf{S} \\
&= \iint_S\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\delta_{mj}\delta_{ik}A_j\frac{\partial n_k^{\mathrm{I}}}{\partial x_i}-\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\delta_{mk}\delta_{ij}A_j\frac{\partial n_k^{\mathrm{I}}}{\partial x_i}\right)\cdot d\mathbf{S} \\
&= \iint_S\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\boldsymbol{\delta}_jA_j\frac{\partial n_i^{\mathrm{I}}}{\partial x_i}-\sum_{i=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_kA_i\frac{\partial n_k^{\mathrm{I}}}{\partial x_i}\right)\cdot d\mathbf{S} \\
&= \iint_S\left[\left(\sum_{i=1}^{3}\frac{\partial n_i^{\mathrm{I}}}{\partial x_i}\right)\sum_{j=1}^{3}\boldsymbol{\delta}_jA_j-\sum_{i=1}^{3}A_i\frac{\partial}{\partial x_i}\left(\sum_{k=1}^{3}\boldsymbol{\delta}_kn_k^{\mathrm{I}}\right)\right]\cdot d\mathbf{S} \\
&= \iint_S\left[(\nabla\cdot\mathbf{n}^{\mathrm{I}})\mathbf{A}-\mathbf{A}\cdot\nabla\mathbf{n}^{\mathrm{I}}\right]\cdot d\mathbf{S} \\
&= \iint_S[(\nabla\cdot\mathbf{n}^{\mathrm{I}})\mathbf{A}-\mathbf{A}\cdot\nabla\mathbf{n}^{\mathrm{I}}]\cdot(\mathbf{n}^{\mathrm{I}}\,dS) \\
&= \iint_S[(\nabla\cdot\mathbf{n}^{\mathrm{I}})\mathbf{A}\cdot\mathbf{n}^{\mathrm{I}}-\mathbf{A}\cdot\nabla\mathbf{n}^{\mathrm{I}}\cdot\mathbf{n}^{\mathrm{I}}]\,dS \\
&= \iint_S\mathbf{A}\cdot[(\nabla\cdot\mathbf{n}^{\mathrm{I}})\mathbf{n}^{\mathrm{I}}-\nabla\mathbf{n}^{\mathrm{I}}\cdot\mathbf{n}^{\mathrm{I}}]\,dS \\
&= \mathbf{A}\cdot\iint_S[(\nabla\cdot\mathbf{n}^{\mathrm{I}})\mathbf{n}^{\mathrm{I}}-\nabla\mathbf{n}^{\mathrm{I}}\cdot\mathbf{n}^{\mathrm{I}}]\,dS
\end{aligned}
$$

Simplify this second term in the integrand.

$$
\begin{aligned}
\nabla\mathbf{n}^{\mathrm{I}}\cdot\mathbf{n}^{\mathrm{I}} &= \left[\left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\left(\sum_{j=1}^{3}\boldsymbol{\delta}_jn_j^{\mathrm{I}}\right)\right]\cdot\left(\sum_{k=1}^{3}\boldsymbol{\delta}_kn_k^{\mathrm{I}}\right) \\
&= \left(\sum_{i=1}^{3}\sum_{j=1}^{3}\boldsymbol{\delta}_i\boldsymbol{\delta}_j\frac{\partial n_j^{\mathrm{I}}}{\partial x_i}\right)\cdot\left(\sum_{k=1}^{3}\boldsymbol{\delta}_kn_k^{\mathrm{I}}\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_i(\boldsymbol{\delta}_j\cdot\boldsymbol{\delta}_k)\frac{\partial n_j^{\mathrm{I}}}{\partial x_i}n_k^{\mathrm{I}}
\end{aligned}
$$

Because $\mathbf{n}^{\mathrm{I}}$ is a unit vector and has a magnitude of 1, the second term in the integrand vanishes.

$$\begin{aligned}
\nabla\mathbf{n}^{\mathrm{I}}\cdot\mathbf{n}^{\mathrm{I}} &= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_i\delta_{jk}\frac{\partial n_j^{\mathrm{I}}}{\partial x_i}n_k^{\mathrm{I}} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\boldsymbol{\delta}_i\frac{\partial n_j^{\mathrm{I}}}{\partial x_i}n_j^{\mathrm{I}} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\boldsymbol{\delta}_i\frac{1}{2}\frac{\partial}{\partial x_i}(n_j^{\mathrm{I}})^2 \\
&= \frac{1}{2}\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\left[\sum_{j=1}^{3}(n_j^{\mathrm{I}})^2\right] \\
&= \frac{1}{2}\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}(1) \\
&= \mathbf{0}
\end{aligned}$$

As a result,

$$\begin{aligned}
\mathbf{A}\cdot\oint_C(\mathbf{n}^{\mathrm{I}}\times\mathbf{t})\,dt &= \mathbf{A}\cdot\iint_S[(\nabla\cdot\mathbf{n}^{\mathrm{I}})\mathbf{n}^{\mathrm{I}} - \cancel{\nabla\mathbf{n}^{\mathrm{I}}\cdot\mathbf{n}^{\mathrm{I}}}]\,dS \\
&= \mathbf{A}\cdot\iint_S\mathbf{n}^{\mathrm{I}}(\nabla\cdot\mathbf{n}^{\mathrm{I}})\,dS.
\end{aligned}$$

It then follows that

$$\oint_C(\mathbf{n}^{\mathrm{I}}\times\mathbf{t})\,dt = \iint_S\mathbf{n}^{\mathrm{I}}(\nabla\cdot\mathbf{n}^{\mathrm{I}})\,dS,$$

and equation (1) becomes

$$\iint_S[\mathbf{n}^{\mathrm{I}}(p^{\mathrm{I}}-p^{\mathrm{II}}) + \mathbf{n}^{\mathrm{I}}\cdot(\boldsymbol{\tau}^{\mathrm{I}}-\boldsymbol{\tau}^{\mathrm{II}})]\,dS = \sigma\iint_S\mathbf{n}^{\mathrm{I}}(\nabla\cdot\mathbf{n}^{\mathrm{I}})\,dS.$$

Bring σ back inside the integral.

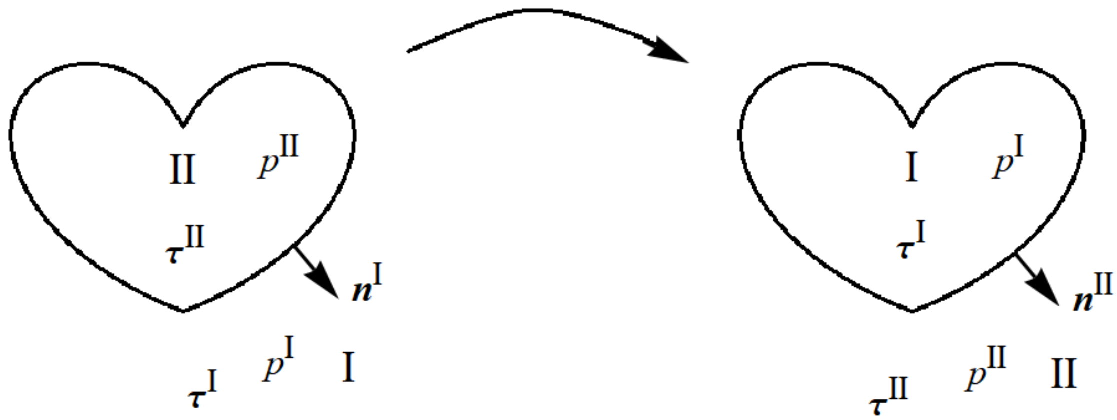
$$\iint_S[\mathbf{n}^{\mathrm{I}}(p^{\mathrm{I}}-p^{\mathrm{II}}) + \mathbf{n}^{\mathrm{I}}\cdot(\boldsymbol{\tau}^{\mathrm{I}}-\boldsymbol{\tau}^{\mathrm{II}})]\,dS = \iint_S\mathbf{n}^{\mathrm{I}}(\nabla\cdot\mathbf{n}^{\mathrm{I}})\sigma\,dS$$

Because the boundary that separates the two immiscible fluids is arbitrary, the surface integrals may be removed.

$$\boxed{\mathbf{n}^{\mathrm{I}}(p^{\mathrm{I}}-p^{\mathrm{II}}) + \mathbf{n}^{\mathrm{I}}\cdot(\boldsymbol{\tau}^{\mathrm{I}}-\boldsymbol{\tau}^{\mathrm{II}}) = \mathbf{n}^{\mathrm{I}}(\nabla\cdot\mathbf{n}^{\mathrm{I}})\sigma} \tag{2}$$

This equation is for the case that phase II is inside the boundary and phase I is outside of it.

But what if it's the other way around?

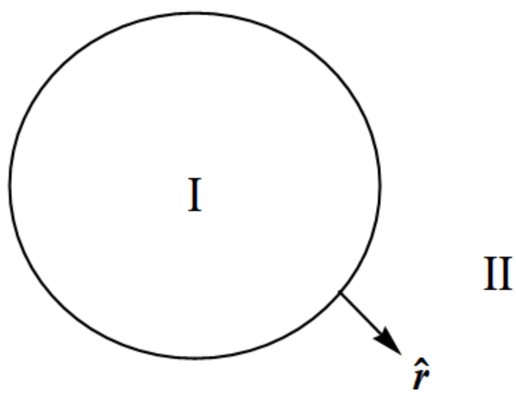


If phase I is inside and phase II is outside, then replace the I superscripts with II and replace the II superscripts with I to get the new boundary condition.

$$\boxed{\mathbf{n}^{\text{II}}(p^{\text{II}} - p^{\text{I}}) + \mathbf{n}^{\text{II}} \cdot (\boldsymbol{\tau}^{\text{II}} - \boldsymbol{\tau}^{\text{I}}) = \mathbf{n}^{\text{II}}(\nabla \cdot \mathbf{n}^{\text{II}})\sigma} \tag{3}$$

Part (a)

Here we have a spherical droplet in a medium. Let phase I be inside the droplet and let phase II be outside of it. Then equation (3) applies. The outward unit normal vector is $\hat{\mathbf{r}}$ if the origin of the coordinate system is at the sphere's center.



Set $\mathbf{n}^{\text{II}} = \hat{\mathbf{r}}$ in equation (3).

$$\hat{\mathbf{r}}(p^{\text{II}} - p^{\text{I}}) + \hat{\mathbf{r}} \cdot (\boldsymbol{\tau}^{\text{II}} - \boldsymbol{\tau}^{\text{I}}) = \hat{\mathbf{r}}(\nabla \cdot \hat{\mathbf{r}})\sigma$$

Use formula (A) on page 836 to calculate the divergence of $\hat{\mathbf{r}}$ in spherical coordinates (r, θ, ϕ). This divergence is to be evaluated at the boundary $r = R$.

$$\begin{aligned}
\hat{\mathbf{r}}(p^{\text{II}} - p^{\text{I}}) + \hat{\mathbf{r}} \cdot (\boldsymbol{\tau}^{\text{II}} - \boldsymbol{\tau}^{\text{I}}) &= \hat{\mathbf{r}} \left[\frac{1}{r^2} \frac{d}{dr}(r^2) \right] \bigg|_{r=R} \sigma \\
&= \hat{\mathbf{r}} \left(\frac{2}{r} \right) \bigg|_{r=R} \sigma \\
&= \hat{\mathbf{r}} \frac{2\sigma}{R}
\end{aligned}$$

The second term on the left side expands as follows in spherical coordinates.

$$\begin{aligned}
\hat{\mathbf{r}}\cdot(\boldsymbol{\tau}^{\mathrm{II}}-\boldsymbol{\tau}^{\mathrm{I}}) &= \boldsymbol{\delta}_r\cdot[\boldsymbol{\delta}_r\boldsymbol{\delta}_r(\tau_{rr}^{\mathrm{II}}-\tau_{rr}^{\mathrm{I}})+\boldsymbol{\delta}_r\boldsymbol{\delta}_\theta(\tau_{r\theta}^{\mathrm{II}}-\tau_{r\theta}^{\mathrm{I}})+\boldsymbol{\delta}_r\boldsymbol{\delta}_\phi(\tau_{r\phi}^{\mathrm{II}}-\tau_{r\phi}^{\mathrm{I}}) \\
&\qquad +\boldsymbol{\delta}_\theta\boldsymbol{\delta}_r(\tau_{\theta r}^{\mathrm{II}}-\tau_{\theta r}^{\mathrm{I}})+\boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta(\tau_{\theta\theta}^{\mathrm{II}}-\tau_{\theta\theta}^{\mathrm{I}})+\boldsymbol{\delta}_\theta\boldsymbol{\delta}_\phi(\tau_{\theta\phi}^{\mathrm{II}}-\tau_{\theta\phi}^{\mathrm{I}}) \\
&\qquad +\boldsymbol{\delta}_\phi\boldsymbol{\delta}_r(\tau_{\phi r}^{\mathrm{II}}-\tau_{\phi r}^{\mathrm{I}})+\boldsymbol{\delta}_\phi\boldsymbol{\delta}_\theta(\tau_{\phi\theta}^{\mathrm{II}}-\tau_{\phi\theta}^{\mathrm{I}})+\boldsymbol{\delta}_\phi\boldsymbol{\delta}_\phi(\tau_{\phi\phi}^{\mathrm{II}}-\tau_{\phi\phi}^{\mathrm{I}})] \\
&= (\boldsymbol{\delta}_r\cdot\boldsymbol{\delta}_r)\boldsymbol{\delta}_r(\tau_{rr}^{\mathrm{II}}-\tau_{rr}^{\mathrm{I}})+(\boldsymbol{\delta}_r\cdot\boldsymbol{\delta}_r)\boldsymbol{\delta}_\theta(\tau_{r\theta}^{\mathrm{II}}-\tau_{r\theta}^{\mathrm{I}})+(\boldsymbol{\delta}_r\cdot\boldsymbol{\delta}_r)\boldsymbol{\delta}_\phi(\tau_{r\phi}^{\mathrm{II}}-\tau_{r\phi}^{\mathrm{I}}) \\
&\qquad +(\boldsymbol{\delta}_r\cdot\boldsymbol{\delta}_\theta)\boldsymbol{\delta}_r(\tau_{\theta r}^{\mathrm{II}}-\tau_{\theta r}^{\mathrm{I}})+(\boldsymbol{\delta}_r\cdot\boldsymbol{\delta}_\theta)\boldsymbol{\delta}_\theta(\tau_{\theta\theta}^{\mathrm{II}}-\tau_{\theta\theta}^{\mathrm{I}})+(\boldsymbol{\delta}_r\cdot\boldsymbol{\delta}_\theta)\boldsymbol{\delta}_\phi(\tau_{\theta\phi}^{\mathrm{II}}-\tau_{\theta\phi}^{\mathrm{I}}) \\
&\qquad +(\boldsymbol{\delta}_r\cdot\boldsymbol{\delta}_\phi)\boldsymbol{\delta}_r(\tau_{\phi r}^{\mathrm{II}}-\tau_{\phi r}^{\mathrm{I}})+(\boldsymbol{\delta}_r\cdot\boldsymbol{\delta}_\phi)\boldsymbol{\delta}_\theta(\tau_{\phi\theta}^{\mathrm{II}}-\tau_{\phi\theta}^{\mathrm{I}})+(\boldsymbol{\delta}_r\cdot\boldsymbol{\delta}_\phi)\boldsymbol{\delta}_\phi(\tau_{\phi\phi}^{\mathrm{II}}-\tau_{\phi\phi}^{\mathrm{I}}) \\
&= \boldsymbol{\delta}_r(\tau_{rr}^{\mathrm{II}}-\tau_{rr}^{\mathrm{I}})+\boldsymbol{\delta}_\theta(\tau_{r\theta}^{\mathrm{II}}-\tau_{r\theta}^{\mathrm{I}})+\boldsymbol{\delta}_\phi(\tau_{r\phi}^{\mathrm{II}}-\tau_{r\phi}^{\mathrm{I}})
\end{aligned}$$

So the boundary condition becomes

$$\hat{\mathbf{r}}(p^{\mathrm{II}}-p^{\mathrm{I}})+\hat{\mathbf{r}}(\tau_{rr}^{\mathrm{II}}-\tau_{rr}^{\mathrm{I}})+\hat{\boldsymbol{\theta}}(\tau_{r\theta}^{\mathrm{II}}-\tau_{r\theta}^{\mathrm{I}})+\hat{\boldsymbol{\phi}}(\tau_{r\phi}^{\mathrm{II}}-\tau_{r\phi}^{\mathrm{I}})=\hat{\mathbf{r}}\frac{2\sigma}{R}.$$

Dot both sides by $\hat{\mathbf{r}}$.

$$(p^{\mathrm{II}}-p^{\mathrm{I}})+(\tau_{rr}^{\mathrm{II}}-\tau_{rr}^{\mathrm{I}})=\frac{2\sigma}{R}$$

Since there is no interfacial viscosity, $\tau_{rr}^{\mathrm{I}}=\tau_{rr}^{\mathrm{II}}$. Therefore, for a spherical droplet (phase I) of radius R and surface tension σ in a medium (phase II),

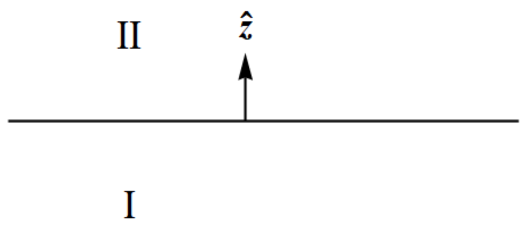
$$\boxed{p^{\mathrm{I}}-p^{\mathrm{II}}=-\frac{2\sigma}{R}.}$$

R and σ are both positive, so the pressure in the droplet is less than that in the medium.

$$p^{\mathrm{I}}-p^{\mathrm{II}}=-\frac{2\sigma}{R}<0$$

$$p^{\mathrm{I}}<p^{\mathrm{II}}$$

Suppose now that there's a flat horizontal boundary. Let phase I be on the bottom and let phase II be on top. Assume that phase I is inside the boundary so that the outward normal unit vector is $\hat{\mathbf{z}}$. Then equation (3) applies once again.



Substitute $\mathbf{n}^{\mathrm{II}} = \hat{\mathbf{z}}$ in equation (3).

$$\hat{\mathbf{z}}(p^{\mathrm{II}} - p^{\mathrm{I}}) + \hat{\mathbf{z}} \cdot (\boldsymbol{\tau}^{\mathrm{II}} - \boldsymbol{\tau}^{\mathrm{I}}) = \hat{\mathbf{z}}(\nabla \cdot \hat{\mathbf{z}})\sigma$$

Use formula (A) on page 832 to calculate the divergence in Cartesian coordinates. This divergence is to be evaluated at the boundary $z = z_0$.

$$\begin{aligned}\hat{\mathbf{z}}(p^{\mathrm{II}} - p^{\mathrm{I}}) + \hat{\mathbf{z}} \cdot (\boldsymbol{\tau}^{\mathrm{II}} - \boldsymbol{\tau}^{\mathrm{I}}) &= \hat{\mathbf{z}} \left[\frac{d}{dz}(1)\right]\bigg|_{z=z_0} \sigma \\ &= \mathbf{0}\end{aligned}$$

The second term on the left side expands as follows in Cartesian coordinates.

$$\begin{aligned}\hat{\mathbf{z}} \cdot (\boldsymbol{\tau}^{\mathrm{II}} - \boldsymbol{\tau}^{\mathrm{I}}) &= \boldsymbol{\delta}_z \cdot [\boldsymbol{\delta}_x\boldsymbol{\delta}_x(\tau^{\mathrm{II}}_{xx} - \tau^{\mathrm{I}}_{xx}) + \boldsymbol{\delta}_x\boldsymbol{\delta}_y(\tau^{\mathrm{II}}_{xy} - \tau^{\mathrm{I}}_{xy}) + \boldsymbol{\delta}_x\boldsymbol{\delta}_z(\tau^{\mathrm{II}}_{xz} - \tau^{\mathrm{I}}_{xz}) \\ &\quad + \boldsymbol{\delta}_y\boldsymbol{\delta}_x(\tau^{\mathrm{II}}_{yx} - \tau^{\mathrm{I}}_{yx}) + \boldsymbol{\delta}_y\boldsymbol{\delta}_y(\tau^{\mathrm{II}}_{yy} - \tau^{\mathrm{I}}_{yy}) + \boldsymbol{\delta}_y\boldsymbol{\delta}_z(\tau^{\mathrm{II}}_{yz} - \tau^{\mathrm{I}}_{yz}) \\ &\quad + \boldsymbol{\delta}_z\boldsymbol{\delta}_x(\tau^{\mathrm{II}}_{zx} - \tau^{\mathrm{I}}_{zx}) + \boldsymbol{\delta}_z\boldsymbol{\delta}_y(\tau^{\mathrm{II}}_{zy} - \tau^{\mathrm{I}}_{zy}) + \boldsymbol{\delta}_z\boldsymbol{\delta}_z(\tau^{\mathrm{II}}_{zz} - \tau^{\mathrm{I}}_{zz})] \\ &= (\boldsymbol{\delta}_z \cdot \boldsymbol{\delta}_x)\boldsymbol{\delta}_x(\tau^{\mathrm{II}}_{xx} - \tau^{\mathrm{I}}_{xx}) + (\boldsymbol{\delta}_z \cdot \boldsymbol{\delta}_x)\boldsymbol{\delta}_y(\tau^{\mathrm{II}}_{xy} - \tau^{\mathrm{I}}_{xy}) + (\boldsymbol{\delta}_z \cdot \boldsymbol{\delta}_x)\boldsymbol{\delta}_z(\tau^{\mathrm{II}}_{xz} - \tau^{\mathrm{I}}_{xz}) \\ &\quad + (\boldsymbol{\delta}_z \cdot \boldsymbol{\delta}_y)\boldsymbol{\delta}_x(\tau^{\mathrm{II}}_{yx} - \tau^{\mathrm{I}}_{yx}) + (\boldsymbol{\delta}_z \cdot \boldsymbol{\delta}_y)\boldsymbol{\delta}_y(\tau^{\mathrm{II}}_{yy} - \tau^{\mathrm{I}}_{yy}) + (\boldsymbol{\delta}_z \cdot \boldsymbol{\delta}_y)\boldsymbol{\delta}_z(\tau^{\mathrm{II}}_{yz} - \tau^{\mathrm{I}}_{yz}) \\ &\quad + (\boldsymbol{\delta}_z \cdot \boldsymbol{\delta}_z)\boldsymbol{\delta}_x(\tau^{\mathrm{II}}_{zx} - \tau^{\mathrm{I}}_{zx}) + (\boldsymbol{\delta}_z \cdot \boldsymbol{\delta}_z)\boldsymbol{\delta}_y(\tau^{\mathrm{II}}_{zy} - \tau^{\mathrm{I}}_{zy}) + (\boldsymbol{\delta}_z \cdot \boldsymbol{\delta}_z)\boldsymbol{\delta}_z(\tau^{\mathrm{II}}_{zz} - \tau^{\mathrm{I}}_{zz}) \\ &= \boldsymbol{\delta}_x(\tau^{\mathrm{II}}_{zx} - \tau^{\mathrm{I}}_{zx}) + \boldsymbol{\delta}_y(\tau^{\mathrm{II}}_{zy} - \tau^{\mathrm{I}}_{zy}) + \boldsymbol{\delta}_z(\tau^{\mathrm{II}}_{zz} - \tau^{\mathrm{I}}_{zz})\end{aligned}$$

So the boundary condition becomes

$$\hat{\mathbf{z}}(p^{\mathrm{II}} - p^{\mathrm{I}}) + \hat{\mathbf{x}}(\tau^{\mathrm{II}}_{zx} - \tau^{\mathrm{I}}_{zx}) + \hat{\mathbf{y}}(\tau^{\mathrm{II}}_{zy} - \tau^{\mathrm{I}}_{zy}) + \hat{\mathbf{z}}(\tau^{\mathrm{II}}_{zz} - \tau^{\mathrm{I}}_{zz}) = \mathbf{0}.$$

Dot both sides by $\hat{\mathbf{z}}$.

$$(p^{\mathrm{II}} - p^{\mathrm{I}}) + (\tau^{\mathrm{II}}_{zz} - \tau^{\mathrm{I}}_{zz}) = 0$$

Since there is no interfacial viscosity, $\tau^{\mathrm{I}}_{zz} = \tau^{\mathrm{II}}_{zz}$. Therefore, for two media separated by a planar boundary,

$$\boxed{p^{\mathrm{I}} - p^{\mathrm{II}} = 0.}$$

The pressures are identical.

$$p^{\mathrm{I}} = p^{\mathrm{II}}$$

This same result would have been obtained if phase II were assumed to be inside the boundary instead.

Part (b)

Start with Eq. 3C.5-1.

$$\mathbf{n}^{\mathrm{I}}(p^{\mathrm{I}} - p^{\mathrm{II}}) + [\mathbf{n}^{\mathrm{I}} \cdot (\boldsymbol{\tau}^{\mathrm{I}} - \boldsymbol{\tau}^{\mathrm{II}})] = \mathbf{n}^{\mathrm{I}} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \sigma \tag{3C.5-1}$$

Make R_1 and R_2 dimensionless first.

$$\begin{aligned}\mathbf{n}^{\mathrm{I}}(p^{\mathrm{I}} - p^{\mathrm{II}}) + [\mathbf{n}^{\mathrm{I}} \cdot (\boldsymbol{\tau}^{\mathrm{I}} - \boldsymbol{\tau}^{\mathrm{II}})] &= \mathbf{n}^{\mathrm{I}} \left(\frac{l_0}{R_1} + \frac{l_0}{R_2} \right) \frac{\sigma}{l_0} \\ &= \mathbf{n}^{\mathrm{I}} \left(\frac{1}{\frac{R_1}{l_0}} + \frac{1}{\frac{R_2}{l_0}} \right) \frac{\sigma}{l_0} \\ &= \mathbf{n}^{\mathrm{I}} \left(\frac{1}{\breve{R}_1} + \frac{1}{\breve{R}_2} \right) \frac{\sigma}{l_0}\end{aligned}$$

Recall that Newton's generalized law of viscosity gives $\boldsymbol{\tau}$ as

$$\boldsymbol{\tau} = -\mu \left(\nabla \mathbf{v} + (\nabla \mathbf{v})^{\dagger} \right) + \left(\frac{2}{3}\mu - \kappa \right) (\nabla \cdot \mathbf{v}) \boldsymbol{\delta}.$$

For a fluid that is incompressible or of constant density, $\nabla \cdot \mathbf{v} = 0$.

$$\boldsymbol{\tau} = -\mu \left(\nabla \mathbf{v} + (\nabla \mathbf{v})^{\dagger} \right)$$

The combination of velocity gradients in parentheses is a second-order tensor $\dot{\boldsymbol{\gamma}}$, which represents the rate of fluid deformation.

$$\boldsymbol{\tau} = -\mu \dot{\boldsymbol{\gamma}}$$

Substitute this formula into Eq. 3C.5-1.

$$\mathbf{n}^{\mathrm{I}}(p^{\mathrm{I}} - p^{\mathrm{II}}) + [\mathbf{n}^{\mathrm{I}} \cdot (-\mu^{\mathrm{I}} \dot{\boldsymbol{\gamma}}^{\mathrm{I}} + \mu^{\mathrm{II}} \dot{\boldsymbol{\gamma}}^{\mathrm{II}})] = \mathbf{n}^{\mathrm{I}} \left(\frac{1}{\breve{R}_1} + \frac{1}{\breve{R}_2} \right) \frac{\sigma}{l_0}$$

$$\mathbf{n}^{\mathrm{I}}(p^{\mathrm{I}} - p^{\mathrm{II}}) - \mu^{\mathrm{I}} \mathbf{n}^{\mathrm{I}} \cdot \dot{\boldsymbol{\gamma}}^{\mathrm{I}} + \mu^{\mathrm{II}} \mathbf{n}^{\mathrm{I}} \cdot \dot{\boldsymbol{\gamma}}^{\mathrm{II}} = \mathbf{n}^{\mathrm{I}} \left(\frac{1}{\breve{R}_1} + \frac{1}{\breve{R}_2} \right) \frac{\sigma}{l_0}$$

The units of $\dot{\boldsymbol{\gamma}}$ are the same as a velocity gradient: velocity over distance. Make $\dot{\boldsymbol{\gamma}}$ dimensionless now.

$$\mathbf{n}^{\mathrm{I}}(p^{\mathrm{I}} - p^{\mathrm{II}}) - \frac{v_0 \mu^{\mathrm{I}}}{l_0} \mathbf{n}^{\mathrm{I}} \cdot \left(\frac{l_0}{v_0} \dot{\boldsymbol{\gamma}}^{\mathrm{I}} \right) + \frac{v_0 \mu^{\mathrm{II}}}{l_0} \mathbf{n}^{\mathrm{I}} \cdot \left(\frac{l_0}{v_0} \dot{\boldsymbol{\gamma}}^{\mathrm{II}} \right) = \mathbf{n}^{\mathrm{I}} \left(\frac{1}{\breve{R}_1} + \frac{1}{\breve{R}_2} \right) \frac{\sigma}{l_0}$$

$$\mathbf{n}^{\mathrm{I}}(p^{\mathrm{I}} - p^{\mathrm{II}}) - \frac{v_0 \mu^{\mathrm{I}}}{l_0} \mathbf{n}^{\mathrm{I}} \cdot \breve{\dot{\boldsymbol{\gamma}}}^{\mathrm{I}} + \frac{v_0 \mu^{\mathrm{II}}}{l_0} \mathbf{n}^{\mathrm{I}} \cdot \breve{\dot{\boldsymbol{\gamma}}}^{\mathrm{II}} = \mathbf{n}^{\mathrm{I}} \left(\frac{1}{\breve{R}_1} + \frac{1}{\breve{R}_2} \right) \frac{\sigma}{l_0}$$

Divide both sides $\rho^{\mathrm{I}} v_0^2$ so that the right side and the last two terms on the left have familiar dimensionless quantities.

$$\mathbf{n}^{\mathrm{I}} \frac{p^{\mathrm{I}} - p^{\mathrm{II}}}{\rho^{\mathrm{I}} v_0^2} - \frac{\mu^{\mathrm{I}}}{l_0 v_0 \rho^{\mathrm{I}}} \mathbf{n}^{\mathrm{I}} \cdot \breve{\dot{\boldsymbol{\gamma}}}^{\mathrm{I}} + \frac{\mu^{\mathrm{II}}}{l_0 v_0 \rho^{\mathrm{I}}} \mathbf{n}^{\mathrm{I}} \cdot \breve{\dot{\boldsymbol{\gamma}}}^{\mathrm{II}} = \mathbf{n}^{\mathrm{I}} \left(\frac{1}{\breve{R}_1} + \frac{1}{\breve{R}_2} \right) \frac{\sigma}{l_0 v_0^2 \rho^{\mathrm{I}}}$$

Make it so that ρ^{II} is grouped with μ^{II} instead of ρ^{I}.

$$\mathbf{n}^{\text{I}}\frac{p^{\text{I}}-p^{\text{II}}}{\rho^{\text{I}}v_0^2}-\frac{\mu^{\text{I}}}{l_0v_0\rho^{\text{I}}}\mathbf{n}^{\text{I}}\cdot\breve{\dot{\boldsymbol{\gamma}}}^{\text{I}}+\frac{\mu^{\text{II}}}{l_0v_0\rho^{\text{II}}}\mathbf{n}^{\text{I}}\cdot\breve{\dot{\boldsymbol{\gamma}}}^{\text{II}}\frac{\rho^{\text{II}}}{\rho^{\text{I}}}=\mathbf{n}^{\text{I}}\left(\frac{1}{\breve{R}_1}+\frac{1}{\breve{R}_2}\right)\frac{\sigma}{l_0v_0^2\rho^{\text{I}}}$$

Now make the pressure dimensionless.

$$\mathbf{n}^{\text{I}}\frac{(p^{\text{I}}-p_0)-(p^{\text{II}}-p_0)}{\rho^{\text{I}}v_0^2}-\frac{\mu^{\text{I}}}{l_0v_0\rho^{\text{I}}}\mathbf{n}^{\text{I}}\cdot\breve{\dot{\boldsymbol{\gamma}}}^{\text{I}}+\frac{\mu^{\text{II}}}{l_0v_0\rho^{\text{II}}}\mathbf{n}^{\text{I}}\cdot\breve{\dot{\boldsymbol{\gamma}}}^{\text{II}}\frac{\rho^{\text{II}}}{\rho^{\text{I}}}=\mathbf{n}^{\text{I}}\left(\frac{1}{\breve{R}_1}+\frac{1}{\breve{R}_2}\right)\frac{\sigma}{l_0v_0^2\rho^{\text{I}}}$$

Include $\rho g(h-h_0)$ to turn these pressures in the numerator into modified pressures.

$$\begin{aligned}\mathbf{n}^{\text{I}}\frac{[p^{\text{I}}-p_0+\rho^{\text{I}}g(h-h_0)]-[p^{\text{II}}-p_0+\rho^{\text{II}}g(h-h_0)]}{\rho^{\text{I}}v_0^2}-\mathbf{n}^{\text{I}}\frac{g(h-h_0)}{v_0^2}+\mathbf{n}^{\text{I}}\frac{\rho^{\text{II}}}{\rho^{\text{I}}}\frac{g(h-h_0)}{v_0^2}&\\-\frac{\mu^{\text{I}}}{l_0v_0\rho^{\text{I}}}\mathbf{n}^{\text{I}}\cdot\breve{\dot{\boldsymbol{\gamma}}}^{\text{I}}+\frac{\mu^{\text{II}}}{l_0v_0\rho^{\text{II}}}\mathbf{n}^{\text{I}}\cdot\breve{\dot{\boldsymbol{\gamma}}}^{\text{II}}\frac{\rho^{\text{II}}}{\rho^{\text{I}}}=\mathbf{n}^{\text{I}}\left(\frac{1}{\breve{R}_1}+\frac{1}{\breve{R}_2}\right)\frac{\sigma}{l_0v_0^2\rho^{\text{I}}}&\end{aligned}$$

$$\begin{aligned}\mathbf{n}^{\text{I}}\left[\frac{p^{\text{I}}-p_0+\rho^{\text{I}}g(h-h_0)}{\rho^{\text{I}}v_0^2}-\frac{p^{\text{II}}-p_0+\rho^{\text{II}}g(h-h_0)}{\rho^{\text{I}}v_0^2}\right]-\mathbf{n}^{\text{I}}\frac{gl_0}{v_0^2}\frac{h-h_0}{l_0}+\mathbf{n}^{\text{I}}\frac{\rho^{\text{II}}}{\rho^{\text{I}}}\frac{gl_0}{v_0^2}\frac{h-h_0}{l_0}&\\-\frac{\mu^{\text{I}}}{l_0v_0\rho^{\text{I}}}\mathbf{n}^{\text{I}}\cdot\breve{\dot{\boldsymbol{\gamma}}}^{\text{I}}+\frac{\mu^{\text{II}}}{l_0v_0\rho^{\text{II}}}\mathbf{n}^{\text{I}}\cdot\breve{\dot{\boldsymbol{\gamma}}}^{\text{II}}\frac{\rho^{\text{II}}}{\rho^{\text{I}}}=\mathbf{n}^{\text{I}}\left(\frac{1}{\breve{R}_1}+\frac{1}{\breve{R}_2}\right)\frac{\sigma}{l_0v_0^2\rho^{\text{I}}}&\end{aligned}$$

$$\begin{aligned}\mathbf{n}^{\text{I}}(\breve{\mathscr{P}}^{\text{I}}-\breve{\mathscr{P}}^{\text{II}})+\mathbf{n}^{\text{I}}\left(\frac{\rho^{\text{II}}}{\rho^{\text{I}}}-1\right)\frac{gl_0}{v_0^2}\frac{h-h_0}{l_0}&\\-\frac{\mu^{\text{I}}}{l_0v_0\rho^{\text{I}}}\mathbf{n}^{\text{I}}\cdot\breve{\dot{\boldsymbol{\gamma}}}^{\text{I}}+\frac{\mu^{\text{II}}}{l_0v_0\rho^{\text{II}}}\mathbf{n}^{\text{I}}\cdot\breve{\dot{\boldsymbol{\gamma}}}^{\text{II}}\frac{\rho^{\text{II}}}{\rho^{\text{I}}}=\mathbf{n}^{\text{I}}\left(\frac{1}{\breve{R}_1}+\frac{1}{\breve{R}_2}\right)\frac{\sigma}{l_0v_0^2\rho^{\text{I}}}&\end{aligned}$$

$$\begin{aligned}\mathbf{n}^{\text{I}}(\breve{\mathscr{P}}^{\text{I}}-\breve{\mathscr{P}}^{\text{II}})+\mathbf{n}^{\text{I}}\left(\frac{\rho^{\text{II}}-\rho^{\text{I}}}{\rho^{\text{I}}}\right)\frac{gl_0}{v_0^2}\breve{h}&\\-\frac{\mu^{\text{I}}}{l_0v_0\rho^{\text{I}}}\mathbf{n}^{\text{I}}\cdot\breve{\dot{\boldsymbol{\gamma}}}^{\text{I}}+\frac{\mu^{\text{II}}}{l_0v_0\rho^{\text{II}}}\mathbf{n}^{\text{I}}\cdot\breve{\dot{\boldsymbol{\gamma}}}^{\text{II}}\frac{\rho^{\text{II}}}{\rho^{\text{I}}}=\mathbf{n}^{\text{I}}\left(\frac{1}{\breve{R}_1}+\frac{1}{\breve{R}_2}\right)\frac{\sigma}{l_0v_0^2\rho^{\text{I}}}&\end{aligned}$$

Therefore, the dimensionless boundary condition is

$$\boxed{\begin{aligned}\mathbf{n}^{\text{I}}(\breve{\mathscr{P}}^{\text{I}}-\breve{\mathscr{P}}^{\text{II}})+\mathbf{n}^{\text{I}}\left[\!\left[\frac{\rho^{\text{II}}-\rho^{\text{I}}}{\rho^{\text{I}}}\right]\!\right]\left[\!\left[\frac{gl_0}{v_0^2}\right]\!\right]\breve{h}&\\-\left[\!\left[\frac{\mu^{\text{I}}}{l_0v_0\rho^{\text{I}}}\right]\!\right][\mathbf{n}^{\text{I}}\cdot\breve{\dot{\boldsymbol{\gamma}}}^{\text{I}}]+\left[\!\left[\frac{\mu^{\text{II}}}{l_0v_0\rho^{\text{II}}}\right]\!\right][\mathbf{n}^{\text{I}}\cdot\breve{\dot{\boldsymbol{\gamma}}}^{\text{II}}]\left[\!\left[\frac{\rho^{\text{II}}}{\rho^{\text{I}}}\right]\!\right]=\mathbf{n}^{\text{I}}\left(\frac{1}{\breve{R}_1}+\frac{1}{\breve{R}_2}\right)\left[\!\left[\frac{\sigma}{l_0v_0^2\rho^{\text{I}}}\right]\!\right],&\end{aligned}}$$

where the thick square brackets enclose dimensionless combinations of variables. From page 98:

$$\text{Re}=\left[\!\left[\frac{l_0v_0\rho}{\mu}\right]\!\right]=\text{Reynolds number}\qquad\text{Fr}=\left[\!\left[\frac{v_0^2}{gl_0}\right]\!\right]=\text{Froude number}\qquad\text{We}=\left[\!\left[\frac{\sigma}{l_0v_0^2\rho}\right]\!\right]=\text{Weber number.}$$

Part (c)

Example 3.7-2 considers the flow of oil (phase I) in a large tank that is exposed to the atmosphere (phase II) on top. Because the density of oil is far greater than that of air ($\rho^{\text{I}} \gg \rho^{\text{II}}$), the ratio $\rho^{\text{II}}/\rho^{\text{I}}$ is essentially zero.

$$\mathbf{n}^{\text{I}}(\breve{\mathscr{P}}^{\text{I}} - \breve{\mathscr{P}}^{\text{II}}) + \mathbf{n}^{\text{I}} \underbrace{\left[\!\left[\frac{\rho^{\text{II}} - \rho^{\text{I}}}{\rho^{\text{I}}}\right]\!\right]}_{\approx -1} \left[\!\left[\frac{gl_0}{v_0^2}\right]\!\right] \breve{h} - \left[\!\left[\frac{\mu^{\text{I}}}{l_0 v_0 \rho^{\text{I}}}\right]\!\right] [\mathbf{n}^{\text{I}} \cdot \breve{\dot{\boldsymbol{\gamma}}}^{\text{I}}] + \left[\!\left[\frac{\mu^{\text{II}}}{l_0 v_0 \rho^{\text{II}}}\right]\!\right] [\mathbf{n}^{\text{I}} \cdot \breve{\dot{\boldsymbol{\gamma}}}^{\text{II}}] \underbrace{\left[\!\left[\frac{\rho^{\text{II}}}{\rho^{\text{I}}}\right]\!\right]}_{\approx 0} = \mathbf{n}^{\text{I}} \left(\frac{1}{\breve{R}_1} + \frac{1}{\breve{R}_2}\right) \left[\!\left[\frac{\sigma}{l_0 v_0^2 \rho^{\text{I}}}\right]\!\right]$$

The gas-liquid boundary has an outward normal unit vector $\mathbf{n}^{\text{I}} = \mathbf{n}$.

$$\mathbf{n}(\breve{\mathscr{P}}^{\text{I}} - \breve{\mathscr{P}}^{\text{II}}) - \mathbf{n} \left[\!\left[\frac{gl_0}{v_0^2}\right]\!\right] \breve{h} - \left[\!\left[\frac{\mu^{\text{I}}}{l_0 v_0 \rho^{\text{I}}}\right]\!\right] [\mathbf{n} \cdot \breve{\dot{\boldsymbol{\gamma}}}^{\text{I}}] = \mathbf{n} \left(\frac{1}{\breve{R}_1} + \frac{1}{\breve{R}_2}\right) \left[\!\left[\frac{\sigma}{l_0 v_0^2 \rho^{\text{I}}}\right]\!\right]$$

Comparing the formulas for $\breve{\mathscr{P}}^{\text{I}}$ and $\breve{\mathscr{P}}^{\text{II}}$, the modified pressure for air is negligible compared to that for oil. Not only is $p^{\text{I}} \gg p^{\text{II}}$, but also the ratio $\rho^{\text{II}}/\rho^{\text{I}}$ appears in $\breve{\mathscr{P}}^{\text{II}}$.

$$\mathbf{n}\, \breve{\mathscr{P}}^{\text{I}} - \mathbf{n} \left[\!\left[\frac{gl_0}{v_0^2}\right]\!\right] \breve{h} - \left[\!\left[\frac{\mu^{\text{I}}}{l_0 v_0 \rho^{\text{I}}}\right]\!\right] [\mathbf{n} \cdot \breve{\dot{\boldsymbol{\gamma}}}^{\text{I}}] = \mathbf{n} \left(\frac{1}{\breve{R}_1} + \frac{1}{\breve{R}_2}\right) \left[\!\left[\frac{\sigma}{l_0 v_0^2 \rho^{\text{I}}}\right]\!\right]$$

Since the tank is large and oil is very viscous, the spinning impeller at the bottom is unlikely to make the gas-liquid boundary nonplanar. That means the sum of $(1/\breve{R}_1)$ and $(1/\breve{R}_2)$ is roughly zero.

$$\mathbf{n}\, \breve{\mathscr{P}}^{\text{I}} - \mathbf{n} \left[\!\left[\frac{gl_0}{v_0^2}\right]\!\right] \breve{h} - \left[\!\left[\frac{\mu^{\text{I}}}{l_0 v_0 \rho^{\text{I}}}\right]\!\right] [\mathbf{n} \cdot \breve{\dot{\boldsymbol{\gamma}}}^{\text{I}}] = \mathbf{0}$$

Since only quantities related to the oil are remaining, remove the I superscripts. Also, substitute $\breve{h} = (h - h_0)/l_0$.

$$\mathbf{n}\, \breve{\mathscr{P}} - \mathbf{n} \left[\!\left[\frac{gl_0}{v_0^2}\right]\!\right] \frac{h - h_0}{l_0} - \left[\!\left[\frac{\mu}{l_0 v_0 \rho}\right]\!\right] [\mathbf{n} \cdot \breve{\dot{\boldsymbol{\gamma}}}] = \mathbf{0}$$

$h - h_0$ is a vertical elevation, so replace it with z. Also, use the characteristic quantities, $v_0 = DN$ and $l_0 = D$.

$$\mathbf{n}\, \breve{\mathscr{P}} - \mathbf{n} \left[\!\left[\frac{gD}{D^2 N^2}\right]\!\right] \frac{z}{D} - \left[\!\left[\frac{\mu}{D^2 N \rho}\right]\!\right] [\mathbf{n} \cdot \breve{\dot{\boldsymbol{\gamma}}}] = \mathbf{0}$$

Therefore, substituting $\breve{z} = z/D$,

$$\boxed{\mathbf{n}\, \breve{\mathscr{P}} - \mathbf{n} \left[\!\left[\frac{g}{DN^2}\right]\!\right] \breve{z} - \left[\!\left[\frac{\mu}{D^2 N \rho}\right]\!\right] [\mathbf{n} \cdot \breve{\dot{\boldsymbol{\gamma}}}] = \mathbf{0}.} \qquad (3.7\text{-}36)$$