Problem 3D.3

Alternate form of the equation of motion.[8] Show that, for an incompressible Newtonian fluid with constant viscosity, the equation of motion may be put into the form

$$4\nabla^2 \mathscr{P} = \rho(\boldsymbol{\omega} : \boldsymbol{\omega}^\dagger - \dot{\boldsymbol{\gamma}} : \dot{\boldsymbol{\gamma}}) \tag{3D.3-1}$$

where

$$\dot{\boldsymbol{\gamma}} = \nabla \mathbf{v} + (\nabla \mathbf{v})^\dagger \text{ and } \boldsymbol{\omega} = \nabla \mathbf{v} - (\nabla \mathbf{v})^\dagger \tag{3D.3-2}$$

Do any additional restrictions have to be placed on this result?

Solution

In terms of the substantial derivative, the Navier-Stokes equation is

$$\frac{D}{Dt}\rho \mathbf{v} = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g}.$$

It holds under the assumption that the fluid density ρ and viscosity μ are constant. The force of gravity is conservative, so there exists a potential function Φ such that $m\mathbf{g} = -\nabla \Phi$.

$$\begin{aligned} \frac{D}{Dt}\rho \mathbf{v} &= -\nabla p + \mu \nabla^2 \mathbf{v} - \frac{\rho}{m}\nabla \Phi \\ &= -\nabla \left(p + \frac{\rho}{m}\Phi\right) + \mu \nabla^2 \mathbf{v} \end{aligned}$$

Introduce the modified pressure function,

$$\mathscr{P} = p + \frac{\rho}{m}\Phi = p + \rho g h,$$

to simplify the right side.

$$\frac{D}{Dt}\rho \mathbf{v} = -\nabla \mathscr{P} + \mu \nabla^2 \mathbf{v}$$

This equation can be written in one of two ways,

$$\begin{aligned} \frac{\partial}{\partial t}\rho \mathbf{v} + \mathbf{v} \cdot \nabla \rho \mathbf{v} &= -\nabla \mathscr{P} + \mu \nabla^2 \mathbf{v} \\ \frac{\partial}{\partial t}\rho \mathbf{v} + \nabla \cdot \rho \mathbf{v}\mathbf{v} &= -\nabla \mathscr{P} + \mu \nabla^2 \mathbf{v}. \end{aligned}$$

Take the divergence of both sides of each equation.

$$\begin{aligned} \nabla \cdot \left(\frac{\partial}{\partial t}\rho \mathbf{v} + \mathbf{v} \cdot \nabla \rho \mathbf{v}\right) &= \nabla \cdot \left(-\nabla \mathscr{P} + \mu \nabla^2 \mathbf{v}\right) \\ \nabla \cdot \left(\frac{\partial}{\partial t}\rho \mathbf{v} + \nabla \cdot \rho \mathbf{v}\mathbf{v}\right) &= \nabla \cdot \left(-\nabla \mathscr{P} + \mu \nabla^2 \mathbf{v}\right) \end{aligned}$$

The divergence of a sum is the sum of the divergences.

$$\begin{aligned} \nabla \cdot \left(\frac{\partial}{\partial t}\rho \mathbf{v}\right) + \nabla \cdot (\mathbf{v} \cdot \nabla \rho \mathbf{v}) &= \nabla \cdot (-\nabla \mathscr{P}) + \nabla \cdot (\mu \nabla^2 \mathbf{v}) \\ \nabla \cdot \left(\frac{\partial}{\partial t}\rho \mathbf{v}\right) + \nabla \cdot (\nabla \cdot \rho \mathbf{v}\mathbf{v}) &= \nabla \cdot (-\nabla \mathscr{P}) + \nabla \cdot (\mu \nabla^2 \mathbf{v}) \end{aligned}$$

[8] P. G. Saffman, *Vortex Dynamics*, Cambridge University Press, corrected edition (1995).

Bring the constants in front.

$$\rho\nabla\cdot\left(\frac{\partial}{\partial t}\mathbf{v}\right)+\rho\nabla\cdot(\mathbf{v}\cdot\nabla\mathbf{v})=-\nabla\cdot\nabla\mathscr{P}+\mu\nabla\cdot(\nabla^2\mathbf{v})$$
$$\rho\nabla\cdot\left(\frac{\partial}{\partial t}\mathbf{v}\right)+\rho\nabla\cdot(\nabla\cdot\mathbf{v}\mathbf{v})=-\nabla\cdot\nabla\mathscr{P}+\mu\nabla\cdot(\nabla^2\mathbf{v})$$

Divide both sides of each equation by ρ and use the kinematic viscosity ν for μ/ρ.

$$\nabla\cdot\left(\frac{\partial}{\partial t}\mathbf{v}\right)+\nabla\cdot(\mathbf{v}\cdot\nabla\mathbf{v})=-\frac{1}{\rho}\nabla^2\mathscr{P}+\nu\nabla\cdot(\nabla^2\mathbf{v}) \tag{1}$$
$$\nabla\cdot\left(\frac{\partial}{\partial t}\mathbf{v}\right)+\nabla\cdot(\nabla\cdot\mathbf{v}\mathbf{v})=-\frac{1}{\rho}\nabla^2\mathscr{P}+\nu\nabla\cdot(\nabla^2\mathbf{v}) \tag{2}$$

Examine the first term on the left side.

$$\begin{aligned}
\nabla\cdot\left(\frac{\partial}{\partial t}\mathbf{v}\right) &= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\cdot\left[\frac{\partial}{\partial t}\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j v_j\right)\right] \\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\cdot\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial v_j}{\partial t}\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}(\boldsymbol{\delta}_i\cdot\boldsymbol{\delta}_j)\frac{\partial}{\partial x_i}\frac{\partial v_j}{\partial t} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\delta_{ij}\frac{\partial}{\partial x_i}\frac{\partial v_j}{\partial t} \\
&= \sum_{i=1}^{3}\frac{\partial}{\partial x_i}\frac{\partial v_i}{\partial t} \\
&= \sum_{i=1}^{3}\frac{\partial}{\partial t}\frac{\partial v_i}{\partial x_i} \\
&= \frac{\partial}{\partial t}\left(\sum_{i=1}^{3}\frac{\partial v_i}{\partial x_i}\right) \\
&= \frac{\partial}{\partial t}(\nabla\cdot\mathbf{v})
\end{aligned}$$

Examine the second term on the right side.

$$\begin{aligned}
\nabla\cdot(\nabla^2\mathbf{v}) &= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\cdot\left[\sum_{j=1}^{3}\frac{\partial^2}{\partial x_j^2}\left(\sum_{l=1}^{3}\boldsymbol{\delta}_l v_l\right)\right]\\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\cdot\left(\sum_{j=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_l\frac{\partial^2 v_l}{\partial x_j^2}\right)\\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_i\cdot\boldsymbol{\delta}_l)\frac{\partial}{\partial x_i}\frac{\partial^2 v_l}{\partial x_j^2}\\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{l=1}^{3}\delta_{il}\frac{\partial}{\partial x_i}\frac{\partial^2 v_l}{\partial x_j^2}\\
&= \sum_{j=1}^{3}\sum_{l=1}^{3}\frac{\partial}{\partial x_l}\frac{\partial^2 v_l}{\partial x_j^2}\\
&= \sum_{j=1}^{3}\sum_{l=1}^{3}\frac{\partial^2}{\partial x_j^2}\frac{\partial v_l}{\partial x_l}\\
&= \sum_{j=1}^{3}\frac{\partial^2}{\partial x_j^2}\left(\sum_{l=1}^{3}\frac{\partial v_l}{\partial x_l}\right)\\
&= \nabla^2(\nabla\cdot\mathbf{v})
\end{aligned}$$

Examine the second term on the left side of equation (2).

$$\begin{aligned}
\nabla\cdot(\nabla\cdot\mathbf{vv}) &= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\cdot\left[\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}\right)\cdot\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k v_k\right)\left(\sum_{l=1}^{3}\boldsymbol{\delta}_l v_l\right)\right]\\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\cdot\left[\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}\right)\cdot\left(\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_k\boldsymbol{\delta}_l v_k v_l\right)\right]\\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\cdot\left[\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_j\cdot\boldsymbol{\delta}_k)\boldsymbol{\delta}_l\frac{\partial}{\partial x_j}v_k v_l\right]\\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\cdot\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\delta_{jk}\boldsymbol{\delta}_l\frac{\partial}{\partial x_j}v_k v_l\right)\\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\cdot\left(\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_l\frac{\partial}{\partial x_k}v_k v_l\right)\\
&= \sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_i\cdot\boldsymbol{\delta}_l)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k}v_k v_l
\end{aligned}$$

Continue simplifying the right side.

$$
\begin{aligned}
\nabla\cdot(\nabla\cdot\mathbf{vv}) &= \sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\delta_{il}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k}v_k v_l \\
&= \sum_{i=1}^{3}\sum_{k=1}^{3}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k}v_k v_i \\
&= \sum_{i=1}^{3}\sum_{k=1}^{3}\frac{\partial}{\partial x_i}\left(\frac{\partial v_k}{\partial x_k}v_i + v_k\frac{\partial v_i}{\partial x_k}\right) \\
&= \sum_{i=1}^{3}\sum_{k=1}^{3}\left(\frac{\partial^2 v_k}{\partial x_i \partial x_k}v_i + \frac{\partial v_k}{\partial x_k}\frac{\partial v_i}{\partial x_i} + \frac{\partial v_k}{\partial x_i}\frac{\partial v_i}{\partial x_k} + v_k\frac{\partial^2 v_i}{\partial x_i \partial x_k}\right) \\
&= \sum_{i=1}^{3}\sum_{k=1}^{3}\frac{\partial^2 v_k}{\partial x_i \partial x_k}v_i + \sum_{i=1}^{3}\sum_{k=1}^{3}\frac{\partial v_k}{\partial x_k}\frac{\partial v_i}{\partial x_i} + \sum_{i=1}^{3}\sum_{k=1}^{3}\frac{\partial v_k}{\partial x_i}\frac{\partial v_i}{\partial x_k} + \sum_{i=1}^{3}\sum_{k=1}^{3}v_k\frac{\partial^2 v_i}{\partial x_i \partial x_k} \\
&= \sum_{i=1}^{3}v_i\frac{\partial}{\partial x_i}\left(\sum_{k=1}^{3}\frac{\partial v_k}{\partial x_k}\right) + \left(\sum_{i=1}^{3}\frac{\partial v_i}{\partial x_i}\right)\left(\sum_{k=1}^{3}\frac{\partial v_k}{\partial x_k}\right) + \sum_{i=1}^{3}\sum_{k=1}^{3}\frac{\partial v_k}{\partial x_i}\frac{\partial v_i}{\partial x_k} + \sum_{k=1}^{3}v_k\frac{\partial}{\partial x_k}\left(\sum_{i=1}^{3}\frac{\partial v_i}{\partial x_i}\right) \\
&= \mathbf{v}\cdot\nabla(\nabla\cdot\mathbf{v}) + (\nabla\cdot\mathbf{v})(\nabla\cdot\mathbf{v}) + (\nabla\mathbf{v}):(\nabla\mathbf{v})^\dagger + \mathbf{v}\cdot\nabla(\nabla\cdot\mathbf{v}) \\
&= 2[\mathbf{v}\cdot\nabla(\nabla\cdot\mathbf{v})] + (\nabla\cdot\mathbf{v})^2 + (\nabla\mathbf{v}):(\nabla\mathbf{v})^\dagger
\end{aligned}
$$

Finally, examine the second term on the left side of equation (1).

$$
\begin{aligned}
\nabla\cdot(\mathbf{v}\cdot\nabla\mathbf{v}) &= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\cdot\left[\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j v_j\right)\cdot\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k\frac{\partial}{\partial x_k}\right)\left(\sum_{l=1}^{3}\boldsymbol{\delta}_l v_l\right)\right] \\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\cdot\left[\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j v_j\right)\cdot\left(\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_k\boldsymbol{\delta}_l\frac{\partial v_l}{\partial x_k}\right)\right] \\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\cdot\left[\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_j\cdot\boldsymbol{\delta}_k)\boldsymbol{\delta}_l v_j\frac{\partial v_l}{\partial x_k}\right] \\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\cdot\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\delta_{jk}\boldsymbol{\delta}_l v_j\frac{\partial v_l}{\partial x_k}\right) \\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\cdot\left(\sum_{j=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_l v_j\frac{\partial v_l}{\partial x_j}\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_i\cdot\boldsymbol{\delta}_l)\frac{\partial}{\partial x_i}\left(v_j\frac{\partial v_l}{\partial x_j}\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{l=1}^{3}\delta_{il}\frac{\partial}{\partial x_i}\left(v_j\frac{\partial v_l}{\partial x_j}\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial}{\partial x_i}\left(v_j\frac{\partial v_i}{\partial x_j}\right)
\end{aligned}
$$

Continue simplifying the right side.

$$\begin{aligned}
\nabla \cdot (\mathbf{v} \cdot \nabla \mathbf{v}) &= \sum_{i=1}^{3} \sum_{j=1}^{3} \left(\frac{\partial v_j}{\partial x_i} \frac{\partial v_i}{\partial x_j} + v_j \frac{\partial^2 v_i}{\partial x_i \partial x_j} \right) \\
&= \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial v_j}{\partial x_i} \frac{\partial v_i}{\partial x_j} + \sum_{i=1}^{3} \sum_{j=1}^{3} v_j \frac{\partial^2 v_i}{\partial x_i \partial x_j} \\
&= \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial v_j}{\partial x_i} \frac{\partial v_i}{\partial x_j} + \sum_{j=1}^{3} v_j \frac{\partial}{\partial x_j} \left(\sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i} \right) \\
&= (\nabla \mathbf{v}) \cdot (\nabla \mathbf{v})^\dagger + \mathbf{v} \cdot \nabla (\nabla \cdot \mathbf{v})
\end{aligned}$$

With these results, equations (1) and (2) become

$$\begin{aligned}
\frac{\partial}{\partial t}(\nabla \cdot \mathbf{v}) + (\nabla \mathbf{v}) \cdot (\nabla \mathbf{v})^\dagger + \mathbf{v} \cdot \nabla (\nabla \cdot \mathbf{v}) &= -\frac{1}{\rho} \nabla^2 \mathscr{P} + \nu \nabla^2 (\nabla \cdot \mathbf{v}) \\
\frac{\partial}{\partial t}(\nabla \cdot \mathbf{v}) + 2[\mathbf{v} \cdot \nabla (\nabla \cdot \mathbf{v})] + (\nabla \cdot \mathbf{v})^2 + (\nabla \mathbf{v}) \cdot (\nabla \mathbf{v})^\dagger &= -\frac{1}{\rho} \nabla^2 \mathscr{P} + \nu \nabla^2 (\nabla \cdot \mathbf{v}).
\end{aligned}$$

Because the density ρ is constant (or the fluid is incompressible), the continuity equation reduces to

$$\frac{\partial \rho}{\partial t} = -(\nabla \cdot \rho \mathbf{v}) \quad \rightarrow \quad 0 = -\rho (\nabla \cdot \mathbf{v}) \quad \rightarrow \quad \nabla \cdot \mathbf{v} = 0,$$

which means these last two equations simplify to

$$(\nabla \mathbf{v}) \cdot (\nabla \mathbf{v})^\dagger = -\frac{1}{\rho} \nabla^2 \mathscr{P}. \tag{3}$$

The quantity in parentheses in the final result simplifies as follows.

$$\begin{aligned}
\boldsymbol{\omega} : \boldsymbol{\omega}^\dagger - \dot{\boldsymbol{\gamma}} : \dot{\boldsymbol{\gamma}} &= \left[\nabla \mathbf{v} - (\nabla \mathbf{v})^\dagger \right] : \left[\nabla \mathbf{v} - (\nabla \mathbf{v})^\dagger \right]^\dagger - \left[\nabla \mathbf{v} + (\nabla \mathbf{v})^\dagger \right] : \left[\nabla \mathbf{v} + (\nabla \mathbf{v})^\dagger \right] \\
&= \left\{ \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i} \right) \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j v_j \right) - \left[\left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i} \right) \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j v_j \right) \right]^\dagger \right\} : \left\{ \left(\sum_{k=1}^{3} \boldsymbol{\delta}_k \frac{\partial}{\partial x_k} \right) \left(\sum_{l=1}^{3} \boldsymbol{\delta}_l v_l \right) - \left[\left(\sum_{k=1}^{3} \boldsymbol{\delta}_k \frac{\partial}{\partial x_k} \right) \left(\sum_{l=1}^{3} \boldsymbol{\delta}_l v_l \right) \right]^\dagger \right\}^\dagger \\
&\quad - \left\{ \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i} \right) \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j v_j \right) + \left[\left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i} \right) \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j v_j \right) \right]^\dagger \right\} : \left\{ \left(\sum_{k=1}^{3} \boldsymbol{\delta}_k \frac{\partial}{\partial x_k} \right) \left(\sum_{l=1}^{3} \boldsymbol{\delta}_l v_l \right) + \left[\left(\sum_{k=1}^{3} \boldsymbol{\delta}_k \frac{\partial}{\partial x_k} \right) \left(\sum_{l=1}^{3} \boldsymbol{\delta}_l v_l \right) \right]^\dagger \right\}
\end{aligned}$$

Continue the simplification.

$$
\begin{aligned}
\boldsymbol{\omega} : \boldsymbol{\omega}^\dagger - \dot{\boldsymbol{\gamma}} : \dot{\boldsymbol{\gamma}} &= \left[\left(\sum_{i=1}^{3}\sum_{j=1}^{3} \boldsymbol{\delta}_i\boldsymbol{\delta}_j \frac{\partial v_j}{\partial x_i}\right) - \left(\sum_{i=1}^{3}\sum_{j=1}^{3} \boldsymbol{\delta}_i\boldsymbol{\delta}_j \frac{\partial v_j}{\partial x_i}\right)^\dagger\right] : \left[\left(\sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_k\boldsymbol{\delta}_l \frac{\partial v_l}{\partial x_k}\right) - \left(\sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_k\boldsymbol{\delta}_l \frac{\partial v_l}{\partial x_k}\right)^\dagger\right]^\dagger \\
&\quad - \left[\left(\sum_{i=1}^{3}\sum_{j=1}^{3} \boldsymbol{\delta}_i\boldsymbol{\delta}_j \frac{\partial v_j}{\partial x_i}\right) + \left(\sum_{i=1}^{3}\sum_{j=1}^{3} \boldsymbol{\delta}_i\boldsymbol{\delta}_j \frac{\partial v_j}{\partial x_i}\right)^\dagger\right] : \left[\left(\sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_k\boldsymbol{\delta}_l \frac{\partial v_l}{\partial x_k}\right) + \left(\sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_k\boldsymbol{\delta}_l \frac{\partial v_l}{\partial x_k}\right)^\dagger\right] \\
&= \left[\left(\sum_{i=1}^{3}\sum_{j=1}^{3} \boldsymbol{\delta}_i\boldsymbol{\delta}_j \frac{\partial v_j}{\partial x_i}\right) - \left(\sum_{i=1}^{3}\sum_{j=1}^{3} \boldsymbol{\delta}_i\boldsymbol{\delta}_j \frac{\partial v_i}{\partial x_j}\right)\right] : \left[\left(\sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_k\boldsymbol{\delta}_l \frac{\partial v_l}{\partial x_k}\right) - \left(\sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_k\boldsymbol{\delta}_l \frac{\partial v_k}{\partial x_l}\right)\right]^\dagger \\
&\quad - \left[\left(\sum_{i=1}^{3}\sum_{j=1}^{3} \boldsymbol{\delta}_i\boldsymbol{\delta}_j \frac{\partial v_j}{\partial x_i}\right) + \left(\sum_{i=1}^{3}\sum_{j=1}^{3} \boldsymbol{\delta}_i\boldsymbol{\delta}_j \frac{\partial v_i}{\partial x_j}\right)\right] : \left[\left(\sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_k\boldsymbol{\delta}_l \frac{\partial v_l}{\partial x_k}\right) + \left(\sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_k\boldsymbol{\delta}_l \frac{\partial v_k}{\partial x_l}\right)\right] \\
&= \left[\sum_{i=1}^{3}\sum_{j=1}^{3} \boldsymbol{\delta}_i\boldsymbol{\delta}_j \left(\frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j}\right)\right] : \left[\sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_k\boldsymbol{\delta}_l \left(\frac{\partial v_l}{\partial x_k} - \frac{\partial v_k}{\partial x_l}\right)\right]^\dagger \\
&\quad - \left[\sum_{i=1}^{3}\sum_{j=1}^{3} \boldsymbol{\delta}_i\boldsymbol{\delta}_j \left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right)\right] : \left[\sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_k\boldsymbol{\delta}_l \left(\frac{\partial v_l}{\partial x_k} + \frac{\partial v_k}{\partial x_l}\right)\right] \\
&= \left[\sum_{i=1}^{3}\sum_{j=1}^{3} \boldsymbol{\delta}_i\boldsymbol{\delta}_j \left(\frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j}\right)\right] : \left[\sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_k\boldsymbol{\delta}_l \left(\frac{\partial v_k}{\partial x_l} - \frac{\partial v_l}{\partial x_k}\right)\right] \\
&\quad - \left[\sum_{i=1}^{3}\sum_{j=1}^{3} \boldsymbol{\delta}_i\boldsymbol{\delta}_j \left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right)\right] : \left[\sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_k\boldsymbol{\delta}_l \left(\frac{\partial v_l}{\partial x_k} + \frac{\partial v_k}{\partial x_l}\right)\right] \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} (\boldsymbol{\delta}_i\boldsymbol{\delta}_j : \boldsymbol{\delta}_k\boldsymbol{\delta}_l) \left(\frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j}\right)\left(\frac{\partial v_k}{\partial x_l} - \frac{\partial v_l}{\partial x_k}\right) \\
&\quad - \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} (\boldsymbol{\delta}_i\boldsymbol{\delta}_j : \boldsymbol{\delta}_k\boldsymbol{\delta}_l) \left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right)\left(\frac{\partial v_l}{\partial x_k} + \frac{\partial v_k}{\partial x_l}\right)
\end{aligned}
$$

Continue the simplification.

$$\begin{aligned}
\boldsymbol{\omega} : \boldsymbol{\omega}^\dagger - \dot{\boldsymbol{\gamma}} : \dot{\boldsymbol{\gamma}} &= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_l)(\boldsymbol{\delta}_j \cdot \boldsymbol{\delta}_k)\left(\frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j}\right)\left(\frac{\partial v_k}{\partial x_l} - \frac{\partial v_l}{\partial x_k}\right) \\
&\quad - \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_l)(\boldsymbol{\delta}_j \cdot \boldsymbol{\delta}_k)\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right)\left(\frac{\partial v_l}{\partial x_k} + \frac{\partial v_k}{\partial x_l}\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\delta_{il}\delta_{jk}\left(\frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j}\right)\left(\frac{\partial v_k}{\partial x_l} - \frac{\partial v_l}{\partial x_k}\right) \\
&\quad - \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\delta_{il}\delta_{jk}\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right)\left(\frac{\partial v_l}{\partial x_k} + \frac{\partial v_k}{\partial x_l}\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\delta_{jk}\left(\frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j}\right)\left(\frac{\partial v_k}{\partial x_i} - \frac{\partial v_i}{\partial x_k}\right) \\
&\quad - \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\delta_{jk}\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right)\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\left(\frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j}\right)\left(\frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j}\right) \\
&\quad - \sum_{i=1}^{3}\sum_{j=1}^{3}\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right)\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\left(\frac{\partial v_j}{\partial x_i}\frac{\partial v_j}{\partial x_i} - \frac{\partial v_j}{\partial x_i}\frac{\partial v_i}{\partial x_j} - \frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\right) \\
&\quad - \sum_{i=1}^{3}\sum_{j=1}^{3}\left(\frac{\partial v_j}{\partial x_i}\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\frac{\partial v_i}{\partial x_j} + \frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i}\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial v_j}{\partial x_i}\frac{\partial v_j}{\partial x_i} - \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial v_j}{\partial x_i}\frac{\partial v_i}{\partial x_j} - \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i} + \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial v_i}{\partial x_j}\frac{\partial v_i}{\partial x_j} \\
&\quad - \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial v_j}{\partial x_i}\frac{\partial v_i}{\partial x_j} - \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial v_j}{\partial x_i}\frac{\partial v_j}{\partial x_i} - \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial v_i}{\partial x_j}\frac{\partial v_i}{\partial x_j} - \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i} \\
&= \nabla\mathbf{v} \cdot \nabla\mathbf{v} - (\nabla\mathbf{v}) \cdot (\nabla\mathbf{v})^\dagger - (\nabla\mathbf{v}) \cdot (\nabla\mathbf{v})^\dagger + \nabla\mathbf{v} \cdot \nabla\mathbf{v} \\
&\quad - (\nabla\mathbf{v}) \cdot (\nabla\mathbf{v})^\dagger - \nabla\mathbf{v} \cdot \nabla\mathbf{v} - \nabla\mathbf{v} \cdot \nabla\mathbf{v} - (\nabla\mathbf{v}) \cdot (\nabla\mathbf{v})^\dagger \\
&= -4(\nabla\mathbf{v}) \cdot (\nabla\mathbf{v})^\dagger
\end{aligned}$$

Solve this result for $(\nabla\mathbf{v}) \cdot (\nabla\mathbf{v})^\dagger$ and plug it into equation (3).

$$-\frac{1}{4}(\boldsymbol{\omega} : \boldsymbol{\omega}^\dagger - \dot{\boldsymbol{\gamma}} : \dot{\boldsymbol{\gamma}}) = -\frac{1}{\rho}\nabla^2\mathscr{P}$$

Therefore,

$$4\nabla^2\mathscr{P} = \rho(\boldsymbol{\omega} : \boldsymbol{\omega}^\dagger - \dot{\boldsymbol{\gamma}} : \dot{\boldsymbol{\gamma}}).$$