Problem 4A.2

Velocity near a moving sphere. A sphere of radius $R$ is falling in creeping flow with a terminal velocity $v_\infty$ through a quiescent fluid of viscosity $\mu$. At what horizontal distance from the sphere does the velocity of the fluid fall to 1% of the terminal velocity of the sphere?

Answer: About 37 diameters

Solution

For a falling sphere in creeping flow (also known as Stokes flow), the following spherical coordinate system was considered in §2.6.

The components of velocity of the surrounding fluid were found by means of a stream function in Example 4.2-1 to be

\[
\begin{align*}
v_r &= v_\infty \left[ 1 - \frac{3}{2} \left( \frac{R}{r} \right) + \frac{1}{2} \left( \frac{R}{r} \right)^3 \right] \cos \theta \\
v_\theta &= v_\infty \left[ -1 + \frac{3}{4} \left( \frac{R}{r} \right) + \frac{1}{4} \left( \frac{R}{r} \right)^3 \right] \sin \theta \\
v_\phi &= 0.
\end{align*}
\]

A horizontal distance from the sphere occurs at $\theta = \pi/2$.

\[
\begin{align*}
v_r \left( r, \frac{\pi}{2}, \phi \right) &= 0 \\
v_\theta \left( r, \frac{\pi}{2}, \phi \right) &= v_\infty \left[ -1 + \frac{3}{4} \left( \frac{R}{r} \right) + \frac{1}{4} \left( \frac{R}{r} \right)^3 \right] \\
v_\phi \left( r, \frac{\pi}{2}, \phi \right) &= 0
\end{align*}
\]
The aim in this problem is to find the horizontal distance from the sphere such that the fluid velocity is 0.01\(v_\infty\) downward. Note that because the fluid is assumed not to slip on the sphere’s surface, points on the sphere have a velocity of \(v_\infty\) downward.

\[
\begin{align*}
\text{\textbullet} & \quad \text{\textbullet} \\
0.01v_\infty & \quad 0.99v_\infty
\end{align*}
\]

The previous formulas cannot be applied at the moment because they were derived for a stationary sphere in a fluid flowing upward from the bottom. Add \(v_\infty \hat{z}\) to the velocity at every point to not only make the sphere stationary, but also to introduce a flow from the bottom.

\[
\begin{align*}
\text{\textbullet} & \quad \text{\textbullet} \\
0 & \quad 0 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 
\end{align*}
\]

Set \(v_\theta = -0.99v_\infty\) and solve this equation for \(r\). The minus sign accounts for the fact that the fluid flows around the sphere in the negative \(\theta\)-direction.

\[
v_\infty \left[ -1 + \frac{3}{4} \left( \frac{R}{r} \right) + \frac{1}{4} \left( \frac{R}{r} \right)^3 \right] = -0.99v_\infty
\]

\[
1 - \frac{3}{4} \left( \frac{R}{r} \right) - \frac{1}{4} \left( \frac{R}{r} \right)^3 = 0.99
\]

\[
0.01 - \frac{3}{4} \left( \frac{R}{r} \right) - \frac{1}{4} \left( \frac{R}{r} \right)^3 = 0
\]
Plot this function on the left side versus \( R/r \) and find where it crosses the horizontal axis.

Therefore,

\[
\frac{R}{r} \approx 0.0133325 \quad \rightarrow \quad r \approx 75R \approx 37 \text{ diameters}.
\]
Here the formulas for the pressure and components of velocity for the flow around a sphere will be derived without using a stream function. For a stationary sphere in a fluid flowing upward from the bottom (illustrated in Fig. 2.6-1), the velocity of the surrounding fluid is assumed to have radial and polar components that both vary with \( r \) and \( \theta \).

\[
v = v_r(r, \theta) \hat{r} + v_\theta(r, \theta) \hat{\theta}
\]

In addition, the pressure is assumed to vary with \( r \) and \( \theta \).

\[
p = p(r, \theta)
\]

One boundary condition is obtained from the assumption that no fluid crosses the spherical surface (that is, it’s impermeable), and a second is obtained from the assumption that the fluid does not slip on the spherical surface.

**Boundary Condition 1:** \( v_r(R, \theta) = 0 \)

**Boundary Condition 2:** \( v_\theta(R, \theta) = 0 \)

Another two boundary conditions are obtained from the fact that the flow is symmetric about the line which is collinear with the polar axis.

**Boundary Condition 3:** \( \frac{\partial v_r}{\partial \theta}(r, 0) = 0 \)

**Boundary Condition 4:** \( \frac{\partial v_r}{\partial \theta}(r, \pi) = 0 \)

Another two boundary conditions are obtained from the fact that the flow is entirely radial at \( \theta = 0 \) and \( \theta = \pi \).

**Boundary Condition 5:** \( v_\theta(r, 0) = 0 \)

**Boundary Condition 6:** \( v_\theta(r, \pi) = 0 \)

The equation of continuity results by considering a mass balance over a volume element that the fluid is flowing through. Since the fluid density is assumed to be constant, the equation simplifies to

\[
\nabla \cdot \mathbf{v} = 0.
\]

Expand the left side in spherical coordinates, using the formula in Appendix B.4 on page 846.

\[
\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (v_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} = 0
\]

Multiply both sides by \( r^2 \).

\[
\frac{\partial}{\partial r} (r^2 v_r) + \frac{r}{\sin \theta} \frac{\partial}{\partial \theta} (v_\theta \sin \theta) = 0
\]

The equation of motion results by considering a momentum balance over a volume element that the fluid is flowing through. Assuming the fluid viscosity \( \mu \) is also constant, the equation simplifies to the Navier-Stokes equation.

\[
\frac{\partial}{\partial t} \rho \mathbf{v} + \nabla \cdot \rho \mathbf{v} \mathbf{v} = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g}
\]
As this is a vector equation, it actually represents three scalar equations, one for each variable in the chosen coordinate system. From Appendix B.6 on page 848, the Navier-Stokes equation yields the following three scalar equations in spherical coordinates.

\[
\begin{align*}
\rho \left( \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_r^2 + v_\theta^2}{r} \right) &= -\frac{\partial p}{\partial r} \\
&\quad + \mu \left[ \frac{1}{r^2} \frac{\partial^2}{\partial r^2} (r^2 v_r) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial v_r}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_r}{\partial \phi^2} \right] + \rho g_r \\
\rho \left( \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta - v_\theta^2 \cot \theta}{r} \right) &= -\frac{1}{r} \frac{\partial p}{\partial \theta} \\
&\quad + \mu \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial v_\theta}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial (v_\theta \sin \theta)}{\partial \theta} \right) \\
&\quad \quad + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\theta}{\partial \phi^2} \right] + \frac{2 \frac{\partial v_r}{\partial \theta}}{r \sin \theta} - \frac{2 \cot \theta \frac{\partial v_\phi}{\partial \phi}}{r^2 \sin \theta} + \rho g_\theta \\
\rho \left( \frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r + v_\theta v_\phi \cot \theta}{r} \right) &= -\frac{1}{r \sin \theta} \frac{\partial p}{\partial \phi} \\
&\quad + \mu \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial v_\phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial (v_\phi \sin \theta)}{\partial \theta} \right) \\
&\quad \quad + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\phi}{\partial \phi^2} \right] + \frac{2 \frac{\partial v_r}{\partial \theta}}{r \sin \theta} + \frac{2 \cot \theta \frac{\partial v_\theta}{\partial \phi}}{r^2 \sin \theta} + \rho g_\phi
\end{align*}
\]

All acceleration terms on each left side are neglected because of the creeping flow assumption. The two relevant equations are as follows.

\[
\begin{align*}
0 &= -\frac{\partial p}{\partial r} + \mu \left[ \frac{1}{r^2} \frac{\partial^2}{\partial r^2} (r^2 v_r) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial v_r}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_r}{\partial \phi^2} \right] + \rho g_r \\
0 &= -\frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial v_\theta}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial (v_\theta \sin \theta)}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\theta}{\partial \phi^2} \right] + \rho g_\theta
\end{align*}
\]

The system of equations (1), (2), and (3) can be solved for the three unknowns, \( p \), \( v_r \), and \( v_\theta \). Multiply both sides of equation (3) by \(-r\).

\[
\begin{align*}
0 &= -\frac{\partial p}{\partial r} + \mu \left[ \frac{1}{r^2} \frac{\partial^2}{\partial r^2} (r^2 v_r) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial v_r}{\partial \theta} \right) \right] + \rho g_r \\
0 &= \frac{\partial p}{\partial \theta} + \mu \left[ -\frac{1}{r} \frac{\partial}{\partial r} \left( r^2 \frac{\partial v_\theta}{\partial r} \right) - \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial (v_\theta \sin \theta)}{\partial \theta} \right) - \frac{2 \frac{\partial v_r}{\partial \theta}}{r \sin \theta} \right] - \rho g_\theta r
\end{align*}
\]

In this problem gravity points straight down: \( g = -g\hat{z} \). Write this unit vector in terms of \( \hat{r} \) and \( \hat{\theta} \) by using formula A.6-33 on page 828.

\[
g = -g[(\cos \theta)\hat{r} + (-\sin \theta)\hat{\theta}] = -g(\cos \theta)\hat{r} + g(\sin \theta)\hat{\theta}
\]
We see that \( g_r = -g \cos \theta \) and \( g_\theta = g \sin \theta \). The previous two equations become

\[
0 = -\frac{\partial p}{\partial r} + \mu \left[ \frac{1}{r^2} \frac{\partial^2}{\partial r^2} (r^2 v_r) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial v_r}{\partial \theta} \right) \right] - \rho g \cos \theta
\]

\[
0 = -\frac{\partial p}{\partial \theta} + \mu \left[ -\frac{1}{r} \frac{\partial}{\partial r} \left( r^2 \frac{\partial v_\theta}{\partial r} \right) - \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{\partial}{\partial \theta} (v_\theta \sin \theta) \right) - 2 \frac{\partial v_r}{r \partial \theta} \right] - \rho g \sin \theta.
\]

Combine the first and last terms on the right side of each equation.

\[
0 = \frac{\partial}{\partial r} (p + \rho g r \cos \theta) + \mu \left[ \frac{1}{r^2} \frac{\partial^2}{\partial r^2} (r^2 v_r) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial v_r}{\partial \theta} \right) \right]
\]

\[
0 = \frac{\partial}{\partial \theta} (p + \rho g r \cos \theta) + \mu \left[ -\frac{1}{r} \frac{\partial}{\partial r} \left( r^2 \frac{\partial v_\theta}{\partial r} \right) - \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{\partial}{\partial \theta} (v_\theta \sin \theta) \right) - 2 \frac{\partial v_r}{r \partial \theta} \right]
\]

Introduce the modified pressure function \( \mathcal{P}(r, \theta) = p(r, \theta) + \rho g r \cos \theta \).

\[
0 = -\frac{\partial \mathcal{P}}{\partial r} + \mu \left[ \frac{1}{r^2} \frac{\partial^2}{\partial r^2} (r^2 v_r) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial v_r}{\partial \theta} \right) \right]
\]

\[
0 = \frac{\partial \mathcal{P}}{\partial \theta} + \mu \left[ -\frac{1}{r} \frac{\partial}{\partial r} \left( r^2 \frac{\partial v_\theta}{\partial r} \right) - \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{\partial}{\partial \theta} (v_\theta \sin \theta) \right) - 2 \frac{\partial v_r}{r \partial \theta} \right]
\]

Differentiate both sides of the first equation with respect to \( \theta \), and differentiate both sides of the second equation with respect to \( r \).

\[
0 = -\frac{\partial^2 \mathcal{P}}{\partial \theta^2} + \mu \left[ \frac{1}{r^2} \frac{\partial^2}{\partial r^2} (r^2 v_r) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial v_r}{\partial \theta} \right) \right]
\]

\[
0 = \frac{\partial^2 \mathcal{P}}{\partial r^2} + \mu \left[ -\frac{1}{r} \frac{\partial}{\partial r} \left( r^2 \frac{\partial v_\theta}{\partial r} \right) - \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{\partial}{\partial \theta} (v_\theta \sin \theta) \right) - 2 \frac{\partial v_r}{r \partial \theta} \right]
\]

Add the respective sides of each equation in order to eliminate the modified pressure. The mixed derivatives are equal by Clairaut’s theorem.

\[
0 = \mu \left[ \frac{1}{r^2} \frac{\partial^2}{\partial r^2} (r^2 v_r) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial v_r}{\partial \theta} \right) \right] + \mu \left[ -\frac{1}{r} \frac{\partial}{\partial r} \left( r^2 \frac{\partial v_\theta}{\partial r} \right) - \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{\partial}{\partial \theta} (v_\theta \sin \theta) \right) - 2 \frac{\partial v_r}{r \partial \theta} \right]
\]

Divide both sides by \( \mu \).

\[
0 = \frac{\partial}{\partial \theta} \left[ \frac{1}{r^2} \frac{\partial^2}{\partial r^2} (r^2 v_r) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial v_r}{\partial \theta} \right) \right] + \frac{\partial}{\partial r} \left[ -\frac{1}{r} \frac{\partial}{\partial r} \left( r^2 \frac{\partial v_\theta}{\partial r} \right) - \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{\partial}{\partial \theta} (v_\theta \sin \theta) \right) - 2 \frac{\partial v_r}{r \partial \theta} \right]
\]

Equations (1) and (6) together form a system of two equations for \( v_r \) and \( v_\theta \). Since both PDEs are linear and homogeneous, the method of separation of variables can be applied to get a solution. Assume that \( v_r \) and \( v_\theta \) have product solutions like so: \( v_r = Q(r)T(\theta) \) and \( v_\theta = \xi(r)\Theta(\theta) \). In particular, based on boundary conditions 3 and 4, we hypothesize that \( T(\theta) = \cos \theta \); in addition, based on boundary conditions 5 and 6, we hypothesize that \( \Theta(\theta) = \sin \theta \).

\[
v_r(r, \theta) = Q(r) \cos \theta
\]

\[
v_\theta(r, \theta) = \xi(r) \sin \theta
\]

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Substitute these formulas into equations (1) and (6).

\[ \frac{\partial}{\partial r} [r^2 Q(r) \cos \theta] + \frac{r}{\sin \theta} \frac{\partial}{\partial \theta} [\xi(r) \sin^2 \theta] = 0 \]

\[ \frac{\partial}{\partial \theta} \left[ \frac{1}{r^2} \frac{\partial^2}{\partial r^2} [r^2 Q(r) \cos \theta] + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} [Q(r) \cos \theta] \right) \right] + \frac{\partial}{\partial r} \left[ -\frac{1}{r} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \xi(r) \sin \theta}{\partial r} \right) \right] - \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} [\xi(r) \sin^2 \theta] \right) - \frac{2}{r} \frac{\partial}{\partial \theta} [Q(r) \cos \theta] = 0 \]

Evaluate the derivatives and expand the left sides.

\[ 2rQ(r) \cos \theta + r^2 Q'(r) \cos \theta + 2r\xi(r) \cos \theta = 0 \]

\[ -Q''(r) \sin \theta - Q'(r) \frac{4 \sin \theta}{r} - Q(r) \frac{2 \sin \theta}{r^2} + \frac{Q(r) 2 \sin \theta}{r^2} \]

\[ -r \xi'''(r) \sin \theta - 3r \xi''(r) \sin \theta + \xi'(r) \frac{2 \sin \theta}{r} - \xi(r) \frac{2 \sin \theta}{r^2} + Q'(r) \frac{2 \sin \theta}{r} - Q(r) \frac{2 \sin \theta}{r^2} = 0 \]

Divide both sides of the first equation by \( r \cos \theta \), and divide both sides of the second equation by \( \sin \theta \).

\[ 2Q(r) + rQ'(r) + 2\xi(r) = 0 \]

\[ -Q''(r) - Q'(r) \frac{2}{r} - Q(r) \frac{2}{r^2} - r \xi'''(r) - 3r \xi''(r) + \xi'(r) \frac{2}{r} - \xi(r) \frac{2}{r^2} = 0 \]

Solve the first equation for \( \xi(r) \)

\[ \xi(r) = -Q(r) - \frac{r}{2} Q'(r) \]  

(7)

and then substitute it into the second equation.

\[ -Q''(r) - Q'(r) \frac{2}{r} - Q(r) \frac{2}{r^2} - r \left[ -Q(r) - \frac{r}{2} Q'(r) \right]'' - 3 \left[ -Q(r) - \frac{r}{2} Q'(r) \right]'' \]

\[ + \left[ -Q(r) - \frac{r}{2} Q'(r) \right] \frac{2}{r} - \left[ -Q(r) - \frac{r}{2} Q'(r) \right] \frac{2}{r^2} = 0 \]

Simplify the left side.

\[ \frac{1}{2} r^2 Q''' + 4r Q'' + 4Q' - \frac{4}{r} Q' = 0 \]

Multiply both sides by \( 2r^2 \).

\[ r^4 Q''' + 8r^3 Q'' + 8r^2 Q' - 8r Q' = 0 \]

This is a homogeneous equidimensional ODE, so its solution is of the form \( Q = r^m \).

\[ Q = r^m \rightarrow Q' = m r^{m-1} \rightarrow Q'' = m(m-1) r^{m-2} \rightarrow Q''' = m(m-1)(m-2) r^{m-3} \]

\[ \rightarrow Q''' = m(m-1)(m-2)(m-3) r^{m-4} \]

Substitute these formulas into the ODE.

\[ r^4(m(m-1)(m-2)(m-3) r^{m-4} + 8r^3(m(m-1)(m-2) r^{m-3} + 8r^2(m(m-1) r^{m-2} - 8 r m r^{m-1} = 0 \]

\[ m(m-1)(m-2)(m-3) r^m + 8m(m-1)(m-2) r^m + 8m(m-1) r^m - 8mr^m = 0 \]
Divide both sides by $r^m$.

$$m(m-1)(m-2)(m-3)+8m(m-1)(m-2)+8m(m-1)-8m = 0$$

Expand the left side.

$$m^4 + 2m^3 - 5m^2 - 6m = 0$$

$$m(m+3)(m+1)(m-2) = 0$$

Four solutions to the ODE are $Q = r^{-3}$ and $Q = r^{-1}$ and $Q = r^0 = 1$ and $Q = r^2$. By the principle of superposition, the general solution to the ODE is a linear combination of these four.

$$Q(r) = C_1r^{-3} + C_2r^{-1} + C_3 + C_4r^2$$

Substitute this formula for $Q$ into equation (7) to get $\xi$.

$$\xi(r) = \frac{C_1}{2}r^{-3} - \frac{C_2}{2}r^{-1} - C_3 - 2C_4r^2$$

Therefore, since $v_r(r, \theta) = Q(r) \cos\theta$ and $v_\theta(r, \theta) = \xi(r) \sin\theta$,

$$v_r(r, \theta) = \left(\frac{C_1}{r^3} + \frac{C_2}{r} + C_3 + C_4r^2\right) \cos\theta$$

$$v_\theta(r, \theta) = \left(\frac{C_1}{2r^3} - \frac{C_2}{2r} - C_3 - 2C_4r^2\right) \sin\theta.$$

Now plug these formulas into equations (4) and (5) to get the modified pressure.

$$0 = -\frac{\partial \mathcal{P}}{\partial r} + 2\mu \left(5C_4 - \frac{C_2}{r^3}\right) \cos\theta$$

$$0 = \frac{\partial \mathcal{P}}{\partial \theta} + \mu \left(\frac{C_2}{r^3} + 10C_4r\right) \sin\theta$$

Solve for the derivatives.

$$\frac{\partial \mathcal{P}}{\partial r} = 2\mu \left(5C_4 - \frac{C_2}{r^3}\right) \cos\theta$$

$$\frac{\partial \mathcal{P}}{\partial \theta} = -\mu \left(\frac{C_2}{r^3} + 10C_4r\right) \sin\theta$$

Integrate both sides of the second equation partially with respect to $\theta$ to get $\mathcal{P}$.

$$\mathcal{P}(r, \theta) = \mu \left(\frac{C_2}{r^2} + 10C_4r\right) \cos\theta + f(r)$$

Differentiate both sides with respect to $r$.

$$\frac{\partial \mathcal{P}}{\partial r} = \mu \left(-2\frac{C_2}{r^3} + 10C_4\right) \cos\theta + f'(r)$$

Comparing this to the previous equation for $\partial \mathcal{P}/\partial r$, we see that

$$f'(r) = 0.$$
Integrate both sides with respect to $r$.

$$f(r) = \mathcal{P}_\infty$$

The modified pressure is then

$$\mathcal{P}(r, \theta) = \mu \left( \frac{C_2}{r^2} + 10C_4r \right) \cos \theta + \mathcal{P}_\infty.$$ 

We require that $\mathcal{P} = \mathcal{P}_\infty$ in the limit as $r \to \infty$: $C_4 = 0$.

$$\mathcal{P}(r, \theta) = \mu \frac{C_2}{r^2} \cos \theta + \mathcal{P}_\infty$$

Therefore, since $p(r, \theta) = \mathcal{P}(r, \theta) - \rho gr \cos \theta$,

$$p(r, \theta) = \mathcal{P}_\infty - \rho gr \cos \theta + \mu \frac{C_2}{r^2} \cos \theta.$$ 

$C_4 = 0$, so the velocity components become

$$v_r(r, \theta) = \left( \frac{C_1}{r^3} + \frac{C_2}{r} + C_3 \right) \cos \theta$$

$$v_\theta(r, \theta) = \left( \frac{C_1}{2r^3} - \frac{C_2}{2r} - C_3 \right) \sin \theta.$$ 

Apply boundary conditions 1 and 2 to determine $C_1$ and $C_3$.

$$v_r(R, \theta) = \left( \frac{C_1}{R^3} + \frac{C_2}{R} + C_3 \right) \cos \theta = 0$$

$$v_\theta(R, \theta) = \left( \frac{C_1}{2R^3} - \frac{C_2}{2R} - C_3 \right) \sin \theta = 0$$

Solving this system of equations yields

$$C_1 = -\frac{C_2R^2}{3} \quad \text{and} \quad C_3 = -\frac{2C_2}{3R}.$$ 

As a result, the velocity components become

$$v_r(r, \theta) = \left( \frac{C_2R^2}{3r^3} + \frac{C_2}{r} - \frac{2C_2}{3R} \right) \cos \theta$$

$$v_\theta(r, \theta) = \left( \frac{C_2R^2}{6r^3} - \frac{C_2}{2r} + \frac{2C_2}{3R} \right) \sin \theta,$$

or after simplifying,

$$v_r(r, \theta) = \frac{2C_2}{3R} \left[ 1 - \frac{3}{2} \left( \frac{R}{r} \right) + \frac{1}{2} \left( \frac{R}{r} \right)^3 \right] \cos \theta$$

$$v_\theta(r, \theta) = -\frac{2C_2}{3R} \left[ -1 + \frac{3}{4} \left( \frac{R}{r} \right) + \frac{1}{4} \left( \frac{R}{r} \right)^3 \right] \sin \theta.$$ 

The boxed formulas in §2.6 are obtained by setting $\mathcal{P}_\infty = p_0$, noting that $z = r \cos \theta$, and using one final boundary condition to determine that $-2C_2/(3R) = v_\infty$. 

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