Problem 4A.2

Velocity near a moving sphere. A sphere of radius R is falling in creeping flow with a terminal velocity v_∞ through a quiescent fluid of viscosity μ. At what horizontal distance from the sphere does the velocity of the fluid fall to 1% of the terminal velocity of the sphere?

Answer: About 37 diameters

Solution

For a falling sphere in creeping flow (also known as Stokes flow), the following spherical coordinate system was considered in §2.6.

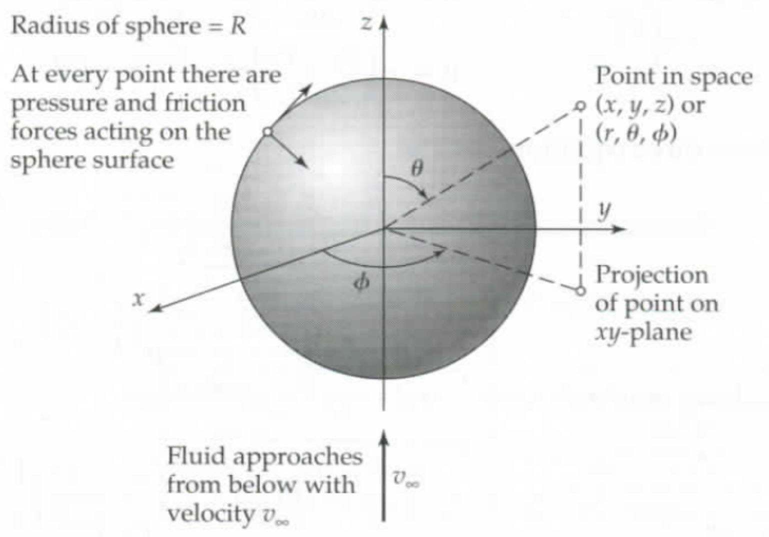


Fig. 2.6-1 Sphere of radius R around which a fluid is flowing. The coordinates r, θ, and ϕ are shown. For more information on spherical coordinates, see Fig. A.8-2.

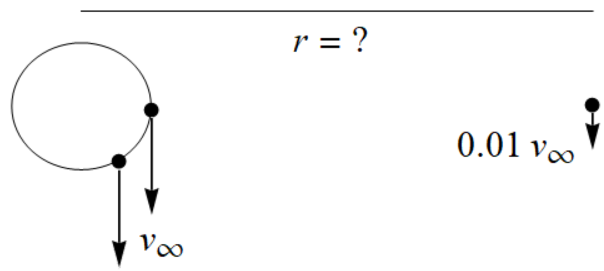
The components of velocity of the surrounding fluid were found by means of a stream function in Example 4.2-1 to be

$$v_r = v_\infty \left[1 - \frac{3}{2}\left(\frac{R}{r}\right) + \frac{1}{2}\left(\frac{R}{r}\right)^3\right]\cos\theta$$
$$v_\theta = v_\infty \left[-1 + \frac{3}{4}\left(\frac{R}{r}\right) + \frac{1}{4}\left(\frac{R}{r}\right)^3\right]\sin\theta$$
$$v_\phi = 0.$$

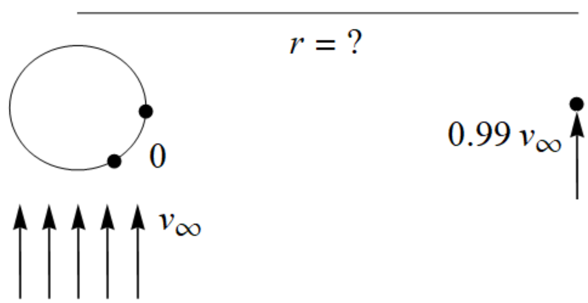
A horizontal distance from the sphere occurs at $\theta = \pi/2$.

$$v_r\left(r, \frac{\pi}{2}, \phi\right) = 0$$
$$v_\theta\left(r, \frac{\pi}{2}, \phi\right) = v_\infty \left[-1 + \frac{3}{4}\left(\frac{R}{r}\right) + \frac{1}{4}\left(\frac{R}{r}\right)^3\right]$$
$$v_\phi\left(r, \frac{\pi}{2}, \phi\right) = 0$$

The aim in this problem is to find the horizontal distance from the sphere such that the fluid velocity is $0.01v_\infty$ downward. Note that because the fluid is assumed not to slip on the sphere's surface, points on the sphere have a velocity of v_∞ downward.



The previous formulas cannot be applied at the moment because they were derived for a stationary sphere in a fluid flowing upward from the bottom. Add $v_\infty\hat{\mathbf{z}}$ to the velocity at every point to not only make the sphere stationary, but also to introduce a flow from the bottom.



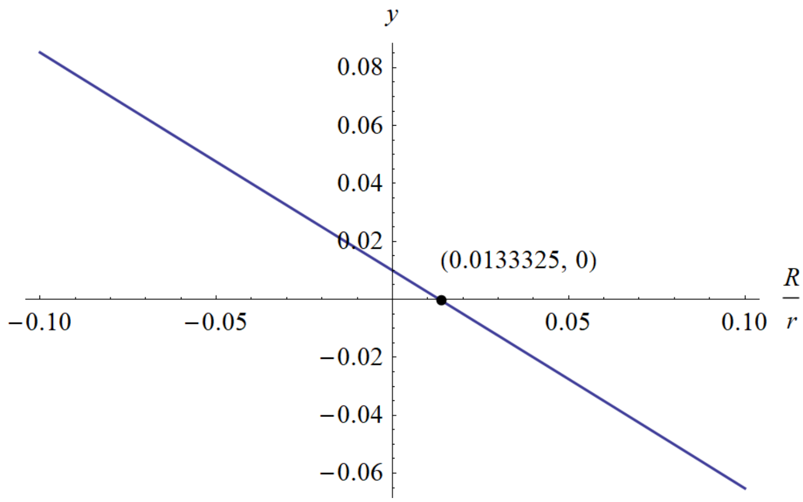
Set $v_\theta = -0.99v_\infty$ and solve this equation for r. The minus sign accounts for the fact that the fluid flows around the sphere in the negative θ-direction.

$$v_\infty\left[-1+\frac{3}{4}\left(\frac{R}{r}\right)+\frac{1}{4}\left(\frac{R}{r}\right)^3\right] = -0.99v_\infty$$

$$1-\frac{3}{4}\left(\frac{R}{r}\right)-\frac{1}{4}\left(\frac{R}{r}\right)^3 = 0.99$$

$$0.01-\frac{3}{4}\left(\frac{R}{r}\right)-\frac{1}{4}\left(\frac{R}{r}\right)^3 = 0$$

Plot this function on the left side versus R/r and find where it crosses the horizontal axis.



Therefore,

$$\frac{R}{r} \approx 0.0133325 \quad \rightarrow \quad r \approx 75R \approx 37 \text{ diameters.}$$

Here the formulas for the pressure and components of velocity for the flow around a sphere will be derived without using a stream function. For a stationary sphere in a fluid flowing upward from the bottom (illustrated in Fig. 2.6-1), the velocity of the surrounding fluid is assumed to have radial and polar components that both vary with r and θ.

$$\mathbf{v} = v_r(r,\theta)\hat{\mathbf{r}} + v_\theta(r,\theta)\hat{\boldsymbol{\theta}}$$

In addition, the pressure is assumed to vary with r and θ.

$$p = p(r,\theta)$$

One boundary condition is obtained from the assumption that no fluid crosses the spherical surface (that is, it's impermeable), and a second is obtained from the assumption that the fluid does not slip on the spherical surface.

$$\begin{aligned} &\text{Boundary Condition 1:} \quad v_r(R,\theta) = 0 \\ &\text{Boundary Condition 2:} \quad v_\theta(R,\theta) = 0 \end{aligned}$$

Another two boundary conditions are obtained from the fact that the flow is symmetric about the line which is collinear with the polar axis.

$$\begin{aligned} &\text{Boundary Condition 3:} \quad \frac{\partial v_r}{\partial \theta}(r,0) = 0 \\ &\text{Boundary Condition 4:} \quad \frac{\partial v_r}{\partial \theta}(r,\pi) = 0 \end{aligned}$$

Another two boundary conditions are obtained from the fact that the flow is entirely radial at $\theta = 0$ and $\theta = \pi$.

$$\begin{aligned} &\text{Boundary Condition 5:} \quad v_\theta(r,0) = 0 \\ &\text{Boundary Condition 6:} \quad v_\theta(r,\pi) = 0 \end{aligned}$$

The equation of continuity results by considering a mass balance over a volume element that the fluid is flowing through. Since the fluid density is assumed to be constant, the equation simplifies to

$$\nabla \cdot \mathbf{v} = 0.$$

Expand the left side in spherical coordinates, using the formula in Appendix B.4 on page 846.

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(v_\theta \sin\theta) + \underbrace{\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}}_{=0} = 0$$

Multiply both sides by r^2.

$$\frac{\partial}{\partial r}(r^2 v_r) + \frac{r}{\sin\theta}\frac{\partial}{\partial \theta}(v_\theta \sin\theta) = 0 \tag{1}$$

The equation of motion results by considering a momentum balance over a volume element that the fluid is flowing through. Assuming the fluid viscosity μ is also constant, the equation simplifies to the Navier-Stokes equation.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla \cdot \rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}$$

As this is a vector equation, it actually represents three scalar equations, one for each variable in the chosen coordinate system. From Appendix B.6 on page 848, the Navier-Stokes equation yields the following three scalar equations in spherical coordinates.

$$\begin{aligned}\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right) &= -\frac{\partial p}{\partial r} \\ &+ \mu\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 v_r) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2}\right] + \rho g_r\end{aligned}$$

$$\begin{aligned}\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta - v_\phi^2\cot\theta}{r}\right) &= -\frac{1}{r}\frac{\partial p}{\partial \theta} \\ &+ \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta)\right)\right. \\ &\left. + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial\phi^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right] + \rho g_\theta\end{aligned}$$

$$\begin{aligned}\rho\left(\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r + v_\theta v_\phi\cot\theta}{r}\right) &= -\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi} \\ &+ \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v_\phi\sin\theta)\right)\right. \\ &\left. + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial\phi^2} + \frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial\phi} + \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\theta}{\partial\phi}\right] + \rho g_\phi\end{aligned}$$

All acceleration terms on each left side are neglected because of the creeping flow assumption. The two relevant equations are as follows.

$$0 = -\frac{\partial p}{\partial r} + \mu\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 v_r) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right) + \overbrace{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2}}^{=0}\right] + \rho g_r \tag{2}$$

$$0 = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta)\right) + \underbrace{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial\phi^2}}_{=0} + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \underbrace{\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}}_{=0}\right] + \rho g_\theta \tag{3}$$

The system of equations (1), (2), and (3) can be solved for the three unknowns, p, v_r, and v_θ. Multiply both sides of equation (3) by $-r$.

$$\begin{aligned}0 &= -\frac{\partial p}{\partial r} + \mu\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 v_r) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right)\right] + \rho g_r \\ 0 &= \frac{\partial p}{\partial \theta} + \mu\left[-\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) - \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta)\right) - \frac{2}{r}\frac{\partial v_r}{\partial\theta}\right] - \rho g_\theta r\end{aligned}$$

In this problem gravity points straight down: $\mathbf{g} = -g\hat{\mathbf{z}}$. Write this unit vector in terms of $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ by using formula A.6-33 on page 828.

$$\mathbf{g} = -g[(\cos\theta)\hat{\mathbf{r}} + (-\sin\theta)\hat{\boldsymbol{\theta}}] = -g(\cos\theta)\hat{\mathbf{r}} + g(\sin\theta)\hat{\boldsymbol{\theta}}$$

We see that $g_r = -g\cos\theta$ and $g_\theta = g\sin\theta$. The previous two equations become

$$0 = -\frac{\partial p}{\partial r} + \mu\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 v_r) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right)\right] - \rho g\cos\theta$$

$$0 = \frac{\partial p}{\partial\theta} + \mu\left[-\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) - \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta)\right) - \frac{2}{r}\frac{\partial v_r}{\partial\theta}\right] - \rho g r\sin\theta.$$

Combine the first and last terms on the right side of each equation.

$$0 = -\frac{\partial}{\partial r}(p + \rho g r\cos\theta) + \mu\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 v_r) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right)\right]$$

$$0 = \frac{\partial}{\partial\theta}(p + \rho g r\cos\theta) + \mu\left[-\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) - \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta)\right) - \frac{2}{r}\frac{\partial v_r}{\partial\theta}\right]$$

Introduce the modified pressure function $\mathscr{P}(r,\theta) = p(r,\theta) + \rho g r\cos\theta$.

$$0 = -\frac{\partial\mathscr{P}}{\partial r} + \mu\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 v_r) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right)\right] \tag{4}$$

$$0 = \frac{\partial\mathscr{P}}{\partial\theta} + \mu\left[-\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) - \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta)\right) - \frac{2}{r}\frac{\partial v_r}{\partial\theta}\right] \tag{5}$$

Differentiate both sides of the first equation with respect to θ, and differentiate both sides of the second equation with respect to r.

$$0 = -\frac{\partial^2\mathscr{P}}{\partial\theta\partial r} + \mu\frac{\partial}{\partial\theta}\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 v_r) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right)\right]$$

$$0 = \frac{\partial^2\mathscr{P}}{\partial r\partial\theta} + \mu\frac{\partial}{\partial r}\left[-\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) - \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta)\right) - \frac{2}{r}\frac{\partial v_r}{\partial\theta}\right]$$

Add the respective sides of each equation in order to eliminate the modified pressure. The mixed derivatives are equal by Clairaut's theorem.

$$0 = \mu\frac{\partial}{\partial\theta}\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 v_r) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right)\right] + \mu\frac{\partial}{\partial r}\left[-\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) - \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta)\right) - \frac{2}{r}\frac{\partial v_r}{\partial\theta}\right]$$

Divide both sides by μ.

$$0 = \frac{\partial}{\partial\theta}\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 v_r) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right)\right] + \frac{\partial}{\partial r}\left[-\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) - \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta)\right) - \frac{2}{r}\frac{\partial v_r}{\partial\theta}\right] \tag{6}$$

Equations (1) and (6) together form a system of two equations for v_r and v_θ. Since both PDEs are linear and homogeneous, the method of separation of variables can be applied to get a solution. Assume that v_r and v_θ have product solutions like so: $v_r = Q(r)T(\theta)$ and $v_\theta = \xi(r)\Theta(\theta)$. In particular, based on boundary conditions 3 and 4, we hypothesize that $T(\theta) = \cos\theta$; in addition, based on boundary conditions 5 and 6, we hypothesize that $\Theta(\theta) = \sin\theta$.

$$v_r(r,\theta) = Q(r)\cos\theta$$

$$v_\theta(r,\theta) = \xi(r)\sin\theta$$

Substitute these formulas into equations (1) and (6).

$$\frac{\partial}{\partial r}[r^2 Q(r)\cos\theta] + \frac{r}{\sin\theta}\frac{\partial}{\partial\theta}[\xi(r)\sin^2\theta] = 0$$

$$\frac{\partial}{\partial\theta}\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}[r^2 Q(r)\cos\theta] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}[Q(r)\cos\theta]\right)\right]$$
$$+\frac{\partial}{\partial r}\left[-\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}[\xi(r)\sin\theta]\right) - \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}[\xi(r)\sin^2\theta]\right) - \frac{2}{r}\frac{\partial}{\partial\theta}[Q(r)\cos\theta]\right] = 0$$

Evaluate the derivatives and expand the left sides.

$$2rQ(r)\cos\theta + r^2 Q'(r)\cos\theta + 2r\xi(r)\cos\theta = 0$$

$$-Q''(r)\sin\theta - Q'(r)\frac{4\sin\theta}{r} - \cancel{Q(r)\frac{2\sin\theta}{r^2}} + \cancel{Q(r)\frac{2\sin\theta}{r^2}}$$
$$-r\xi'''(r)\sin\theta - 3\xi''(r)\sin\theta + \xi'(r)\frac{2\sin\theta}{r} - \xi(r)\frac{2\sin\theta}{r^2} + Q'(r)\frac{2\sin\theta}{r} - Q(r)\frac{2\sin\theta}{r^2} = 0$$

Divide both sides of the first equation by $r\cos\theta$, and divide both sides of the second equation by $\sin\theta$.

$$2Q(r) + rQ'(r) + 2\xi(r) = 0$$
$$-Q''(r) - Q'(r)\frac{2}{r} - Q(r)\frac{2}{r^2} - r\xi'''(r) - 3\xi''(r) + \xi'(r)\frac{2}{r} - \xi(r)\frac{2}{r^2} = 0$$

Solve the first equation for $\xi(r)$

$$\xi(r) = -Q(r) - \frac{r}{2}Q'(r) \tag{7}$$

and then substitute it into the second equation.

$$-Q''(r) - Q'(r)\frac{2}{r} - Q(r)\frac{2}{r^2} - r\left[-Q(r) - \frac{r}{2}Q'(r)\right]''' - 3\left[-Q(r) - \frac{r}{2}Q'(r)\right]''$$
$$+\left[-Q(r) - \frac{r}{2}Q'(r)\right]'\frac{2}{r} - \left[-Q(r) - \frac{r}{2}Q'(r)\right]\frac{2}{r^2} = 0$$

Simplify the left side.

$$\frac{1}{2}r^2 Q'''' + 4rQ''' + 4Q'' - \frac{4}{r}Q' = 0$$

Multiply both sides by $2r^2$.

$$r^4 Q'''' + 8r^3 Q''' + 8r^2 Q'' - 8rQ' = 0$$

This is a homogeneous equidimensional ODE, so its solution is of the form $Q = r^m$.

$$Q = r^m \quad\rightarrow\quad Q' = mr^{m-1} \quad\rightarrow\quad Q'' = m(m-1)r^{m-2} \quad\rightarrow\quad Q''' = m(m-1)(m-2)r^{m-3}$$
$$\rightarrow\quad Q'''' = m(m-1)(m-2)(m-3)r^{m-4}$$

Substitute these formulas into the ODE.

$$r^4 m(m-1)(m-2)(m-3)r^{m-4} + 8r^3 m(m-1)(m-2)r^{m-3} + 8r^2 m(m-1)r^{m-2} - 8rmr^{m-1} = 0$$

$$m(m-1)(m-2)(m-3)r^m + 8m(m-1)(m-2)r^m + 8m(m-1)r^m - 8mr^m = 0$$

Divide both sides by r^m.

$$m(m-1)(m-2)(m-3) + 8m(m-1)(m-2) + 8m(m-1) - 8m = 0$$

Expand the left side.

$$m^4 + 2m^3 - 5m^2 - 6m = 0$$

$$m(m+3)(m+1)(m-2) = 0$$

$$m = \{-3, -1, 0, 2\}$$

Four solutions to the ODE are $Q = r^{-3}$ and $Q = r^{-1}$ and $Q = r^0 = 1$ and $Q = r^2$. By the principle of superposition, the general solution to the ODE is a linear combination of these four.

$$Q(r) = C_1 r^{-3} + C_2 r^{-1} + C_3 + C_4 r^2$$

Substitute this formula for Q into equation (7) to get ξ.

$$\xi(r) = \frac{C_1}{2} r^{-3} - \frac{C_2}{2} r^{-1} - C_3 - 2C_4 r^2$$

Therefore, since $v_r(r,\theta) = Q(r)\cos\theta$ and $v_\theta(r,\theta) = \xi(r)\sin\theta$,

$$v_r(r,\theta) = \left(\frac{C_1}{r^3} + \frac{C_2}{r} + C_3 + C_4 r^2\right)\cos\theta$$

$$v_\theta(r,\theta) = \left(\frac{C_1}{2r^3} - \frac{C_2}{2r} - C_3 - 2C_4 r^2\right)\sin\theta.$$

Now plug these formulas into equations (4) and (5) to get the modified pressure.

$$0 = -\frac{\partial \mathscr{P}}{\partial r} + 2\mu\left(5C_4 - \frac{C_2}{r^3}\right)\cos\theta$$

$$0 = \frac{\partial \mathscr{P}}{\partial \theta} + \mu\left(\frac{C_2}{r^2} + 10C_4 r\right)\sin\theta$$

Solve for the derivatives.

$$\frac{\partial \mathscr{P}}{\partial r} = 2\mu\left(5C_4 - \frac{C_2}{r^3}\right)\cos\theta$$

$$\frac{\partial \mathscr{P}}{\partial \theta} = -\mu\left(\frac{C_2}{r^2} + 10C_4 r\right)\sin\theta$$

Integrate both sides of the second equation partially with respect to θ to get $\mathscr{P}$.

$$\mathscr{P}(r,\theta) = \mu\left(\frac{C_2}{r^2} + 10C_4 r\right)\cos\theta + f(r)$$

Differentiate both sides with respect to r.

$$\frac{\partial \mathscr{P}}{\partial r} = \mu\left(-\frac{2C_2}{r^3} + 10C_4\right)\cos\theta + f'(r)$$

Comparing this to the previous equation for $\partial\mathscr{P}/\partial r$, we see that

$$f'(r) = 0.$$

Integrate both sides with respect to r.

$$f(r) = \mathscr{P}_\infty$$

The modified pressure is then

$$\mathscr{P}(r,\theta) = \mu\left(\frac{C_2}{r^2} + 10C_4 r\right)\cos\theta + \mathscr{P}_\infty.$$

We require that $\mathscr{P} = \mathscr{P}_\infty$ in the limit as $r \to \infty$: $C_4 = 0$.

$$\mathscr{P}(r,\theta) = \mu\frac{C_2}{r^2}\cos\theta + \mathscr{P}_\infty$$

Therefore, since $p(r,\theta) = \mathscr{P}(r,\theta) - \rho g r\cos\theta$,

$$\boxed{p(r,\theta) = \mathscr{P}_\infty - \rho g r\cos\theta + \mu\frac{C_2}{r^2}\cos\theta.}$$

$C_4 = 0$, so the velocity components become

$$\begin{aligned} v_r(r,\theta) &= \left(\frac{C_1}{r^3} + \frac{C_2}{r} + C_3\right)\cos\theta \\ v_\theta(r,\theta) &= \left(\frac{C_1}{2r^3} - \frac{C_2}{2r} - C_3\right)\sin\theta. \end{aligned}$$

Apply boundary conditions 1 and 2 to determine C_1 and C_3.

$$\begin{aligned} v_r(R,\theta) &= \left(\frac{C_1}{R^3} + \frac{C_2}{R} + C_3\right)\cos\theta = 0 \\ v_\theta(R,\theta) &= \left(\frac{C_1}{2R^3} - \frac{C_2}{2R} - C_3\right)\sin\theta = 0 \end{aligned}$$

Solving this system of equations yields

$$C_1 = -\frac{C_2R^2}{3} \quad \text{and} \quad C_3 = -\frac{2C_2}{3R}.$$

As a result, the velocity components become

$$\begin{aligned} v_r(r,\theta) &= \left(-\frac{C_2R^2}{3r^3} + \frac{C_2}{r} - \frac{2C_2}{3R}\right)\cos\theta \\ v_\theta(r,\theta) &= \left(-\frac{C_2R^2}{6r^3} - \frac{C_2}{2r} + \frac{2C_2}{3R}\right)\sin\theta, \end{aligned}$$

or after simplifying,

$$\boxed{\begin{aligned} v_r(r,\theta) &= -\frac{2C_2}{3R}\left[1 - \frac{3}{2}\left(\frac{R}{r}\right) + \frac{1}{2}\left(\frac{R}{r}\right)^3\right]\cos\theta \\ v_\theta(r,\theta) &= -\frac{2C_2}{3R}\left[-1 + \frac{3}{4}\left(\frac{R}{r}\right) + \frac{1}{4}\left(\frac{R}{r}\right)^3\right]\sin\theta. \end{aligned}}$$

The boxed formulas in §2.6 are obtained by setting $\mathscr{P}_\infty = p_0$, noting that $z = r\cos\theta$, and using one final boundary condition to determine that $-2C_2/(3R) = v_\infty$.