Problem 4A.3

Construction of streamlines for the potential flow around a cylinder. Plot the streamlines for the flow around a cylinder using the information in Example 4.3-1 by the following procedure:

(a) Select a value of $\Psi = C$ (that is, select a streamline).

(b) Plot $Y = C + K$ (straight lines parallel to the $X$-axis) and $Y = K(X^2 + Y^2)$ (circles with radius $1/2K$, tangent to the $X$-axis at the origin).

(c) Plot the intersections of the lines and circles that have the same value of $K$.

(d) Join these points to get the streamline for $\Psi = C$.

Then select other values of $C$ and repeat the process until the pattern of streamlines is clear.

[TYPOS: Plot $Y = K - C$ instead. Also, the radius of the circle should be $1/2|K|$.

Solution

According to Example 4.3-1, the stream function in terms of the dimensionless variables, $\Psi$ and $X$ and $Y$, is given by Eq. 4.3-19.

$$\Psi(X, Y) = -Y \left(1 - \frac{1}{X^2 + Y^2}\right) \quad (4.3-19)$$

With a Computer

Below is a graph of $\Psi(X, Y) = -4$, $\Psi(X, Y) = -2$, $\Psi(X, Y) = -1$, $\Psi(X, Y) = 1$, $\Psi(X, Y) = 2$, $\Psi(X, Y) = 4$, and $\Psi(X, Y) = 0$ in red, orange, yellow, green, blue, purple, and black, respectively.
Decrease the spacing between streamlines by plotting $\Psi(X, Y) = -4$, $\Psi(X, Y) = -3.9$, $\Psi(X, Y) = -3.8$, $\Psi(X, Y) = 3.8$, $\Psi(X, Y) = 3.9$, and $\Psi(X, Y) = 4$. 
Highlight the circle $X^2 + Y^2 = 1$ black and ignore the ink within it to obtain a picture of the fluid flow around a cylinder.
Without a Computer

Pretend that we don’t have software sophisticated enough to graph the streamlines \( \Psi(X, Y) = C \). Then we have to tediously construct them one by one.

\[
\Psi(X, Y) = -Y \left(1 - \frac{1}{X^2 + Y^2}\right) = C
\]

The issue is that this is a cubic equation for \( Y \), so the aim is to break it down into simpler functions that we can graph. Consider the case that \( C = 0 \). Then by the zero product theorem,

\[
-Y = 0 \quad \text{or} \quad 1 - \frac{1}{X^2 + Y^2} = 0
\]

\[
Y = 0 \quad \text{or} \quad X^2 + Y^2 = 1.
\]

Now consider the case that \( C \) is nonzero.

\[
\Psi(X, Y) = -Y + \frac{Y}{X^2 + Y^2} = C
\]

\[
\frac{Y}{X^2 + Y^2} = C + Y
\]

The only way a function of \( X \) and \( Y \) can be equal to a function of \( Y \) is if both are equal to some constant \( K \).

\[
\frac{Y}{X^2 + Y^2} = C + Y = K \quad \Rightarrow \quad \begin{cases} 
C + Y = K \\
\frac{Y}{X^2 + Y^2} = K
\end{cases}
\]

The two simpler functions we have to graph then are

\[
Y = K - C \\
Y = K(X^2 + Y^2).
\]
This second function can actually be written as

\[
X^2 + Y^2 - \frac{Y}{K} = 0
\]

\[
X^2 + Y^2 - \frac{Y}{K} + \frac{1}{4K^2} = \frac{1}{4K^2}
\]

\[
X^2 + \left( Y - \frac{1}{2K} \right)^2 = \frac{1}{4K^2},
\]

which is a circle with

Center: \( \left( 0, \frac{1}{2K} \right) \) \quad \text{Radius:} \quad \frac{1}{2|K|}.

**The Red Streamline**

To obtain the red streamline shown in the first figure, set \( C = -4 \)

\[
Y = K + 4
\]

\[
Y = K(X^2 + Y^2).
\]

and plot these functions for the values of \( K \) that they intersect.
Superimpose the graphs.

Finally, connect the dots with a smooth curve to obtain the red streamline.
The Orange Streamline

To obtain the orange streamline shown in the first figure, set $C = -2$

$$Y = K + 2$$
$$Y = K(X^2 + Y^2).$$

and plot these functions for the values of $K$ that they intersect.
Superimpose the graphs.

Finally, connect the dots with a smooth curve to obtain the orange streamline.
The Yellow Streamline

To obtain the yellow streamline shown in the first figure, set $C = -1$

\[
Y = K + 1 \\
Y = K(X^2 + Y^2).
\]

and plot these functions for the values of $K$ that they intersect.
Superimpose the graphs.

Finally, connect the dots with a smooth curve to obtain the yellow streamline.
The Green Streamline

To obtain the green streamline shown in the first figure, set $C = 1$

\[
Y = K - 1
\]
\[
Y = K(X^2 + Y^2)
\]

and plot these functions for the values of $K$ that they intersect.

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Superimpose the graphs.

Finally, connect the dots with a smooth curve to obtain the green streamline.
The Blue Streamline

To obtain the blue streamline shown in the first figure, set \( C = 2 \)

\[
Y = K - 2 \\
Y = K(X^2 + Y^2).
\]

and plot these functions for the values of \( K \) that they intersect.
Superimpose the graphs.

Finally, connect the dots with a smooth curve to obtain the blue streamline.
The Purple Streamline

To obtain the purple streamline shown in the first figure, set $C = 4$

\[
Y = K - 4
\]
\[
Y = K(X^2 + Y^2).
\]

and plot these functions for the values of $K$ that they intersect.

![Graphs showing purple streamline for different values of $K$]
Superimpose the graphs.

Finally, connect the dots with a smooth curve to obtain the purple streamline.