Problem 4B.1

Flow of a fluid with a suddenly applied constant wall stress. In the system studied in Example 4.1-1, let the fluid be at rest before $t = 0$. At time $t = 0$ a constant force is applied to the fluid at the wall in the positive x direction, so that the shear stress τ_{yx} takes on a new constant value τ_0 at $y = 0$ for $t > 0$.

(a) Differentiate Eq. 4.1-1 with respect to y and multiply by $-\mu$ to obtain a partial differential equation for $\tau_{yx}(y, t)$.

(b) Write the boundary and initial conditions for this equation.

(c) Solve using the method in Example 4.1-1 to obtain

$$\frac{\tau_{yx}}{\tau_0} = 1 - \operatorname{erf} \frac{y}{\sqrt{4\nu t}} \tag{4B.1-1}$$

(d) Use the result in (c) to obtain the velocity profile. The following relation[1] will be helpful

$$\int_x^\infty (1 - \operatorname{erf} u)\, du = \frac{1}{\sqrt{\pi}} e^{-x^2} - x(1 - \operatorname{erf} x) \tag{4B.1-2}$$

Solution

The aim in Example 4.1-1 was to find the fluid velocity in response to a wall set into motion with velocity v_0. Here in this problem we will find the fluid velocity in response to a constant shearing force per unit area τ_0 on the $y = 0$ plane in the x-direction. This velocity is assumed to flow only in the x-direction and vary with y and t.

$$\mathbf{v} = v_x(y, t)\hat{\mathbf{x}}$$

At and prior to the time when the shear stress acts, the velocity is zero everywhere.

$$\text{Initial Condition:} \quad v_x(y, t) = 0, \quad t \le 0,\ 0 \le y < \infty$$

Also, far, far away from the $y = 0$ plane, the velocity will always be zero.

$$\text{Boundary Condition:} \quad \lim_{y \to \infty} v_x(y, t) = 0, \quad -\infty < t < \infty$$

The equation of continuity results by considering a mass balance over a volume element that the fluid is flowing through. Assuming the fluid density ρ is constant, the equation simplifies to

$$\nabla \cdot \mathbf{v} = 0. \tag{1}$$

The equation of motion results by considering a momentum balance over a volume element that the fluid is flowing through. Assuming the fluid viscosity μ is also constant, the equation simplifies to the Navier-Stokes equation.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla \cdot \rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g} \tag{2}$$

[1] A useful summary of error functions and their properties can be found in H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, Oxford University Press, 2nd edition (1959), Appendix II.

As this is a vector equation, it actually represents three scalar equations—one for each variable in the chosen coordinate system. Using Cartesian coordinates is the appropriate choice for this problem, so equations (1) and (2) will be expanded in (x, y, z). From Appendix B.4 on page 846, the continuity equation becomes

$$\underbrace{\frac{\partial v_x}{\partial x}}_{=0} + \underbrace{\frac{\partial v_y}{\partial y}}_{=0} + \underbrace{\frac{\partial v_z}{\partial z}}_{=0} = 0,$$

which doesn't tell us anything. From Appendix B.6 on page 848, the Navier-Stokes equation yields the following three scalar equations in Cartesian coordinates. (Gravity is assumed to be entirely in the y-direction, and no pressure gradients exist in the x- and z-directions.)

$$\rho\left(\frac{\partial v_x}{\partial t} + \underbrace{v_x\frac{\partial v_x}{\partial x}}_{=0} + \underbrace{v_y\frac{\partial v_x}{\partial y}}_{=0} + \underbrace{v_z\frac{\partial v_x}{\partial z}}_{=0}\right) = \underbrace{-\frac{\partial p}{\partial x}}_{=0} + \mu\left[\underbrace{\frac{\partial^2 v_x}{\partial x^2}}_{=0} + \frac{\partial^2 v_x}{\partial y^2} + \underbrace{\frac{\partial^2 v_x}{\partial z^2}}_{=0}\right] + \underbrace{\rho g_x}_{=0}$$

$$\rho\left(\underbrace{\frac{\partial v_y}{\partial t}}_{=0} + \underbrace{v_x\frac{\partial v_y}{\partial x}}_{=0} + \underbrace{v_y\frac{\partial v_y}{\partial y}}_{=0} + \underbrace{v_z\frac{\partial v_y}{\partial z}}_{=0}\right) = -\frac{\partial p}{\partial y} + \mu\left[\underbrace{\frac{\partial^2 v_y}{\partial x^2}}_{=0} + \underbrace{\frac{\partial^2 v_y}{\partial y^2}}_{=0} + \underbrace{\frac{\partial^2 v_y}{\partial z^2}}_{=0}\right] + \rho g_y$$

$$\rho\left(\underbrace{\frac{\partial v_z}{\partial t}}_{=0} + \underbrace{v_x\frac{\partial v_z}{\partial x}}_{=0} + \underbrace{v_y\frac{\partial v_z}{\partial y}}_{=0} + \underbrace{v_z\frac{\partial v_z}{\partial z}}_{=0}\right) = \underbrace{-\frac{\partial p}{\partial z}}_{=0} + \mu\left[\underbrace{\frac{\partial^2 v_z}{\partial x^2}}_{=0} + \underbrace{\frac{\partial^2 v_z}{\partial y^2}}_{=0} + \underbrace{\frac{\partial^2 v_z}{\partial z^2}}_{=0}\right] + \underbrace{\rho g_z}_{=0}$$

The relevant equation for the velocity is the x-equation, which has simplified considerably from the assumption that $\mathbf{v} = v_x(y, t)\hat{\boldsymbol{x}}$.

$$\rho\frac{\partial v_x}{\partial t} = \mu\frac{\partial^2 v_x}{\partial y^2}$$

Divide both sides by ρ and use the kinematic viscosity $\nu = \mu/\rho$.

$$\frac{\partial v_x}{\partial t} = \nu\frac{\partial^2 v_x}{\partial y^2}$$

Differentiate both sides with respect to y.

$$\frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial t}\right) = \frac{\partial}{\partial y}\left(\nu\frac{\partial^2 v_x}{\partial y^2}\right)$$

Bring ν in front and switch the order of differentiation using Clairaut's theorem.

$$\frac{\partial}{\partial t}\left(\frac{\partial v_x}{\partial y}\right) = \nu\frac{\partial^2}{\partial y^2}\left(\frac{\partial v_x}{\partial y}\right)$$

Multiply both sides by $-\mu$.

$$-\mu\frac{\partial}{\partial t}\left(\frac{\partial v_x}{\partial y}\right) = -\mu\nu\frac{\partial^2}{\partial y^2}\left(\frac{\partial v_x}{\partial y}\right)$$

Since μ is a constant, it can be brought inside the derivatives.

$$\frac{\partial}{\partial t}\left(-\mu\frac{\partial v_x}{\partial y}\right) = \nu\frac{\partial^2}{\partial y^2}\left(-\mu\frac{\partial v_x}{\partial y}\right)$$

According to Appendix B.1 on page 843, τ_{yx} is

$$\tau_{yx} = -\mu\left[\underbrace{\frac{\partial v_y}{\partial x}}_{=\,0} + \frac{\partial v_x}{\partial y}\right] = -\mu\frac{\partial v_x}{\partial y}.$$

Substituting this formula into the previous equation gives a PDE for the shear stress $\tau_{yx} = \tau_{yx}(y,t)$.

$$\frac{\partial \tau_{yx}}{\partial t} = \nu\frac{\partial^2 \tau_{yx}}{\partial y^2}, \quad 0 < y < \infty,\ t > 0$$

A constant shear stress τ_0 is applied on the $y = 0$ plane in the x-direction for all $t \geq 0$.

$$\text{Boundary Condition 1:} \quad \tau_{yx}(0,t) = \tau_0, \quad t \geq 0$$

Far, far away from the $y = 0$ plane, the shear stress will always be zero.

$$\text{Boundary Condition 2:} \quad \lim_{y\to\infty} \tau(y,t) = 0, \quad -\infty < t < \infty$$

Prior to $t = 0$, the shear stress is zero everywhere.

$$\text{Initial Condition:} \quad \tau(y,t) = 0, \quad t < 0,\ 0 \leq y < \infty$$

In order to make the dependent variable dimensionless, divide both sides of the PDE for τ_{yx} by τ_0.

$$\frac{1}{\tau_0}\frac{\partial \tau_{yx}}{\partial t} = \frac{\nu}{\tau_0}\frac{\partial^2 \tau_{yx}}{\partial y^2}$$

Bring $1/\tau_0$ inside the derivatives.

$$\frac{\partial}{\partial t}\left(\frac{\tau_{yx}}{\tau_0}\right) = \nu\frac{\partial^2}{\partial y^2}\left(\frac{\tau_{yx}}{\tau_0}\right)$$

Let $\phi(y,t) = \tau_{yx}/\tau_0$ be the dimensionless shear stress.

$$\frac{\partial \phi}{\partial t} = \nu\frac{\partial^2 \phi}{\partial y^2}, \quad 0 < y < \infty,\ t > 0$$

To solve this PDE on the half-line, make use of the similarity argument: Since ϕ is dimensionless, the general solution must be a function of the remaining variables in some dimensionless grouping. y has units of distance, t has units of time, and ν has units of distance^2/time. The combination of variables,

$$\eta = \frac{y}{\sqrt{4\nu t}},$$

is most convenient for the diffusion equation, so

$$\phi(y,t) = f\left(\frac{y}{\sqrt{4\nu t}}\right).$$

Find the partial derivatives of ϕ

$$\frac{\partial \phi}{\partial t} = f'\left(\frac{y}{\sqrt{4\nu t}}\right)\cdot\frac{-y}{4\sqrt{\nu t^3}}$$

$$\frac{\partial \phi}{\partial y} = f'\left(\frac{y}{\sqrt{4\nu t}}\right)\cdot\frac{1}{\sqrt{4\nu t}}$$

$$\frac{\partial \phi}{\partial y} = f''\left(\frac{y}{\sqrt{4\nu t}}\right)\cdot\frac{1}{4\nu t}$$

and substitute them into the PDE.

$$-\frac{y}{4\sqrt{\nu t^3}}f' = \nu\left(\frac{1}{4\nu t}f''\right)$$

$$\frac{1}{4t}f'' + \frac{y}{4\sqrt{\nu t^3}}f' = 0$$

$$f'' + \frac{y}{\sqrt{\nu t}}f' = 0$$

$$f'' + 2\frac{y}{\sqrt{4\nu t}}f' = 0$$

$$f'' + 2\eta f' = 0$$

This is a first-order linear ODE for f, so it can be solved with an integrating factor I.

$$I = \exp\left(\int^{\eta} 2s\, ds\right) = e^{\eta^2}$$

Multiply both sides of the ODE by I.

$$e^{\eta^2}f'' + 2\eta e^{\eta^2}f' = 0$$

The left side can be written as $d/d\eta(If')$ by the product rule.

$$\frac{d}{d\eta}(e^{\eta^2}f') = 0$$

Integrate both sides with respect to η.

$$e^{\eta^2}f' = C_1$$

Divide both sides by e^{η^2}.

$$f' = C_1e^{-\eta^2}$$

Integrate both sides with respect to η once more.

$$f(\eta) = \int^{\eta} C_1e^{-s^2}\, ds + C_2$$

The lower limit of integration is arbitrary, so it will be set to zero.

$$f(\eta) = C_1\int_0^{\eta} e^{-s^2}\, ds + C_2$$

This means the general solution is

$$\phi(y,t) = f\left(\frac{y}{\sqrt{4\nu t}}\right) = C_1\int_0^{y/\sqrt{4\nu t}} e^{-s^2}\, ds + C_2.$$

Obtain the boundary conditions for ϕ from those for τ_{yx}.

$$\tau_{yx}(0,t) = \tau_0 \quad \rightarrow \quad \frac{\tau_{yx}(0,t)}{\tau_0} = 1 \quad \rightarrow \quad \phi(0,t) = 1$$

$$\lim_{y\to\infty}\tau(y,t) = 0 \quad \rightarrow \quad \lim_{y\to\infty}\frac{\tau(y,t)}{\tau_0} = 0 \quad \rightarrow \quad \lim_{y\to\infty}\phi(y,t) = 0$$

Apply them now to determine C_1 and C_2.

$$\phi(0,t) = C_2 = 1$$

$$\lim_{y\to\infty} \phi(y,t) = C_1 \int_0^\infty e^{-s^2}\,ds + C_2 = C_1 \frac{\sqrt{\pi}}{2} + C_2 = 0$$

Solving this system of equations yields $C_1 = -2/\sqrt{\pi}$ and $C_2 = 1$.

$$\phi(y,t) = -\frac{2}{\pi} \int_0^{y/\sqrt{4\nu t}} e^{-s^2}\,ds + 1.$$

The error function is a special function which is defined as

$$\operatorname{erf} z = \frac{2}{\pi} \int_0^{y/\sqrt{4\nu t}} e^{-s^2}\,ds,$$

so the solution can be written as

$$\phi(y,t) = 1 - \operatorname{erf}\left(\frac{y}{\sqrt{4\nu t}}\right).$$

Therefore, since $\phi = \tau_{yx}/\tau_0$,

$$\boxed{\frac{\tau_{yx}}{\tau_0} = 1 - \operatorname{erf}\left(\frac{y}{\sqrt{4\nu t}}\right).}$$

Multiply both sides by τ_0.

$$\tau_{yx}(y,t) = \tau_0 \left[1 - \operatorname{erf}\left(\frac{y}{\sqrt{4\nu t}}\right)\right]$$

Change back to the velocity.

$$-\mu \frac{\partial v_x}{\partial y} = \tau_0 \left[1 - \operatorname{erf}\left(\frac{y}{\sqrt{4\nu t}}\right)\right]$$

Divide both sides by $-\mu$.

$$\frac{\partial v_x}{\partial y} = -\frac{\tau_0}{\mu} \left[1 - \operatorname{erf}\left(\frac{y}{\sqrt{4\nu t}}\right)\right]$$

To get v_x, integrate both sides partially with respect to y.

$$\int^y \frac{\partial v_x}{\partial y}\bigg|_{y=r} dr = \int^y -\frac{\tau_0}{\mu} \left[1 - \operatorname{erf}\left(\frac{r}{\sqrt{4\nu t}}\right)\right] dr + h(t)$$

$$v_x(y,t) = -\frac{\tau_0}{\mu} \int^y \left[1 - \operatorname{erf}\left(\frac{r}{\sqrt{4\nu t}}\right)\right] dr + h(t)$$

Because $h(t)$ is an arbitrary function, the lower limit of integration is arbitrary. Set it to ∞.

$$v_x(y,t) = -\frac{\tau_0}{\mu} \int_\infty^y \left[1 - \operatorname{erf}\left(\frac{r}{\sqrt{4\nu t}}\right)\right] dr + h(t)$$

Make the substitution,

$$u = \frac{r}{\sqrt{4\nu t}}$$

$$du = \frac{dr}{\sqrt{4\nu t}} \quad \to \quad \sqrt{4\nu t}\,du = dr.$$

Then the integral becomes

$$\begin{aligned}
v_x(y,t) &= -\frac{\tau_0}{\mu}\int_{\infty}^{y/\sqrt{4\nu t}} (1-\operatorname{erf} u)\,(\sqrt{4\nu t}\,du) + h(t) \\
&= \frac{\tau_0}{\mu}\sqrt{4\nu t}\int_{y/\sqrt{4\nu t}}^{\infty} (1-\operatorname{erf} u)\,du + h(t) \\
&= \frac{\tau_0}{\mu}\sqrt{4\nu t}\left[\frac{1}{\sqrt{\pi}}e^{-u^2} - u(1-\operatorname{erf} u)\right]\Bigg|^{y/\sqrt{4\nu t}} + h(t) \\
&= \frac{\tau_0}{\mu}\sqrt{4\nu t}\left[\frac{1}{\sqrt{\pi}}e^{-y^2/(4\nu t)} - \frac{y}{\sqrt{4\nu t}}\left(1-\operatorname{erf}\frac{y}{\sqrt{4\nu t}}\right)\right] + h(t) \\
&= \frac{\tau_0}{\mu}\left[\sqrt{\frac{4\nu t}{\pi}}e^{-y^2/(4\nu t)} - y\left(1-\operatorname{erf}\frac{y}{\sqrt{4\nu t}}\right)\right] + h(t).
\end{aligned}$$

There is another special function known as the complementary error function which is defined as

$$\operatorname{erfc} z = 1 - \operatorname{erf} z,$$

so the previous equation further simplifies to

$$v_x(y,t) = \frac{\tau_0}{\mu}\left[\sqrt{\frac{4\nu t}{\pi}}\exp\left(-\frac{y^2}{4\nu t}\right) - y\operatorname{erfc}\frac{y}{\sqrt{4\nu t}}\right] + h(t).$$

Whether $y \to \infty$ or $t \to 0$, the initial and boundary conditions for v_x imply that $v_x = h(t) = 0$. Therefore,

$$\boxed{v_x(y,t) = \frac{\tau_0}{\mu}\left[\sqrt{\frac{4\nu t}{\pi}}\exp\left(-\frac{y^2}{4\nu t}\right) - y\operatorname{erfc}\frac{y}{\sqrt{4\nu t}}\right], \quad t \geq 0.}$$

Below are plots of v_x versus y for $\tau_0 = \mu = \rho = \nu = 1$ at various values of t. The times, $t = 0$, $t = 0.05$, $t = 0.22$, $t = 0.5$, $t = 1$, $t = 4$, and $t = 10$, correspond to the graphs in red, orange, yellow, green, blue, purple, and gray, respectively. There is no steady-state velocity distribution here, unlike in Example 4.1-1 for a wall moving at speed v_0. The fluid velocity keeps increasing.