Problem 4B.4

Use of the vorticity equation.

(a) Work Problem 2B.3 using the y-component of the vorticity equation (Eq. 3D.2-1) and the following boundary conditions: $x = \pm B$, $v_z = 0$ and at $x = 0$, $v_z = v_{z,\max}$. Show that this leads to

$$v_z = v_{z,\max}[1 - (x/B)^2] \tag{4B.4-1}$$

Then obtain the pressure distribution from the z-component of the equation of motion.

(b) Work Problem 3B.6(b) using the vorticity equation, with the following boundary conditions: at $r = R$, $v_z = 0$ and at $r = \kappa R$, $v_z = v_0$. In addition an integral condition is needed to state that there is no net flow in the z direction. Find the pressure distribution in the system.

(c) Work the following problems using the vorticity equation: 2B.6, 2B.7, 3B.1, 3B.10, 3B.16.

Solution

As discussed in Problem 3D.2, the vorticity equation,

$$\frac{\partial}{\partial t}\mathbf{w} + \mathbf{v}\cdot\nabla\mathbf{w} = \nu\nabla^2\mathbf{w} + \mathbf{w}\cdot\nabla\mathbf{v}, \tag{1}$$

results from taking the curl of both sides of the Navier-Stokes equation. $\mathbf{w} = \nabla\times\mathbf{v}$ is known as the vorticity.

Problem 2B.3

This problem considers the flow of fluid through two parallel plates.

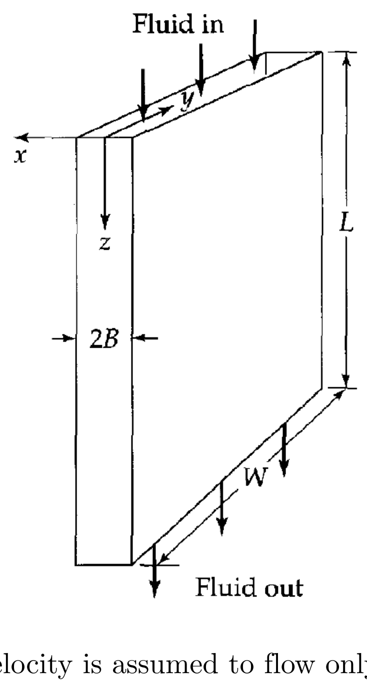

Fig. 2B.3 Flow through a slit, with $B << W << L$.

The velocity is assumed to flow only in the z-direction and vary in the x-direction.

$$\mathbf{v} = v_z(x)\hat{\mathbf{z}}$$

As a result, the vorticity only has a y-component.

$$\mathbf{w} = \nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & v_z \end{vmatrix} = -\frac{dv_z}{dx}\hat{\mathbf{y}}$$

Substitute this into equation (1).

$$\underbrace{\frac{\partial}{\partial t}\left(-\frac{dv_z}{dx}\hat{\mathbf{y}}\right)}_{=\mathbf{0}} + \mathbf{v}\cdot\nabla\left(-\frac{dv_z}{dx}\hat{\mathbf{y}}\right) = \nu\nabla^2\left(-\frac{dv_z}{dx}\hat{\mathbf{y}}\right) + \left(-\frac{dv_z}{dx}\hat{\mathbf{y}}\right)\cdot\nabla\mathbf{v}$$

The components of $\mathbf{v}\cdot\nabla\mathbf{w}$ and $\mathbf{w}\cdot\nabla\mathbf{v}$ in Cartesian coordinates are given in the formulas at the bottom of page 832. Since only v_z and w_y are nonzero, only the last term in equation (Q) remains in the former and only the second term in equation (R) remains in the latter.

$$\underbrace{v_z(x)\frac{\partial}{\partial z}\left(-\frac{dv_z}{dx}\right)\hat{\mathbf{y}}}_{=\mathbf{0}} = \nu\nabla^2\left(-\frac{dv_z}{dx}\hat{\mathbf{y}}\right) + \underbrace{\left(-\frac{dv_z}{dx}\right)\frac{\partial v_z}{\partial y}\hat{\mathbf{y}}}_{=\mathbf{0}}$$

Because v_z is only a function of x, these terms are zero anyway. Lastly, use equation (N) on page 832 to write out the Laplacian.

$$\mathbf{0} = \nu\left[\frac{\partial^2}{\partial x^2}\left(-\frac{dv_z}{dx}\right) + \underbrace{\frac{\partial^2}{\partial y^2}\left(-\frac{dv_z}{dx}\right)}_{=0} + \underbrace{\frac{\partial^2}{\partial z^2}\left(-\frac{dv_z}{dx}\right)}_{=0}\right]\hat{\mathbf{y}}$$

Dot both sides by $\hat{\mathbf{y}}$ to get a scalar equation.

$$0 = -\nu\frac{d^3 v_z}{dx^3}$$

Divide both sides by $-\nu$.

$$\frac{d^3 v_z}{dx^3} = 0$$

Integrate both sides with respect to x.

$$\frac{d^2 v_z}{dx^2} = C_1$$

Integrate both sides with respect to x once more.

$$\frac{dv_z}{dx} = C_1 x + C_2$$

Integrate both sides with respect to x once more.

$$v_z(x) = \frac{1}{2}C_1 x^2 + C_2 x + C_3$$

Because three boundary conditions are provided, C_1 and C_2 and C_3 can be determined.

Apply the boundary conditions now.

$$\begin{aligned} v_z(0) &= C_3 = v_{z,\max} \\ v_z(-B) &= \frac{1}{2}C_1B^2 - C_2B + C_3 \\ v_z(B) &= \frac{1}{2}C_1B^2 + C_2B + C_3 \end{aligned}$$

Solving this system of equations yields

$$C_1 = -\frac{2v_{z,\max}}{B^2} \quad \text{and} \quad C_2 = 0 \quad \text{and} \quad C_3 = v_{z,\max}.$$

Therefore,

$$\begin{aligned} v_z(x) &= -\frac{v_{z,\max}}{B^2}x^2 + v_{z,\max} \\ &= v_{z,\max}\left(-\frac{x^2}{B^2} + 1\right) \\ &= v_{z,\max}\left[1 - \left(\frac{x}{B}\right)^2\right]. \end{aligned}$$

To get the pressure distribution, the Navier-Stokes equation is needed.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla\cdot\rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}$$

From Appendix B.6 on page 848, it yields the following three scalar equations in Cartesian coordinates.

$$\begin{aligned} \rho\left(\underbrace{\frac{\partial v_x}{\partial t}}_{=0} + \underbrace{v_x\frac{\partial v_x}{\partial x}}_{=0} + \underbrace{v_y\frac{\partial v_x}{\partial y}}_{=0} + \underbrace{v_z\frac{\partial v_x}{\partial z}}_{=0}\right) &= -\frac{\partial p}{\partial x} + \mu\left[\underbrace{\frac{\partial^2 v_x}{\partial x^2}}_{=0} + \underbrace{\frac{\partial^2 v_x}{\partial y^2}}_{=0} + \underbrace{\frac{\partial^2 v_x}{\partial z^2}}_{=0}\right] + \rho g_x \\ \rho\left(\underbrace{\frac{\partial v_y}{\partial t}}_{=0} + \underbrace{v_x\frac{\partial v_y}{\partial x}}_{=0} + \underbrace{v_y\frac{\partial v_y}{\partial y}}_{=0} + \underbrace{v_z\frac{\partial v_y}{\partial z}}_{=0}\right) &= -\frac{\partial p}{\partial y} + \mu\left[\underbrace{\frac{\partial^2 v_y}{\partial x^2}}_{=0} + \underbrace{\frac{\partial^2 v_y}{\partial y^2}}_{=0} + \underbrace{\frac{\partial^2 v_y}{\partial z^2}}_{=0}\right] + \rho g_y \\ \rho\left(\underbrace{\frac{\partial v_z}{\partial t}}_{=0} + \underbrace{v_x\frac{\partial v_z}{\partial x}}_{=0} + \underbrace{v_y\frac{\partial v_z}{\partial y}}_{=0} + \underbrace{v_z\frac{\partial v_z}{\partial z}}_{=0}\right) &= -\frac{\partial p}{\partial z} + \mu\left[\frac{\partial^2 v_z}{\partial x^2} + \underbrace{\frac{\partial^2 v_z}{\partial y^2}}_{=0} + \underbrace{\frac{\partial^2 v_z}{\partial z^2}}_{=0}\right] + \rho g_z \end{aligned}$$

Gravity points down in the z-direction, so $\mathbf{g} = g\hat{\mathbf{z}}$, which means $g_x = 0$, $g_y = 0$, and $g_z = g$.

$$\begin{aligned} 0 &= -\frac{\partial p}{\partial x} \\ 0 &= -\frac{\partial p}{\partial y} \\ 0 &= -\frac{\partial p}{\partial z} + \mu\frac{d^2 v_z}{dx^2} + \rho g \end{aligned}$$

These first two equations imply that p is neither a function of x nor a function of y: $p = p(z)$.

$$0 = -\frac{dp}{dz} + \mu\frac{d^2 v_z}{dx^2} + \rho g$$

Solve for dp/dz and substitute d^2v_z/dx^2.

$$\frac{dp}{dz} = \mu\left(-\frac{2v_{z,\max}}{B^2}\right) + \rho g$$

Integrate both sides with respect to z, using p_0 (the pressure at $z = 0$) as the integration constant.

$$\begin{aligned} p(z) &= \left(-\frac{2\mu v_{z,\max}}{B^2} + \rho g\right) z + p_0 \\ &= -\frac{2\mu v_{z,\max}}{B^2} z + \rho g z + p_0 \end{aligned}$$

Bring $\rho g z$ to the left side.

$$p(z) - \rho g z = -\frac{2\mu v_{z,\max}}{B^2} z + p_0 - \rho g(0)$$

Define the modified pressure here to be as it was in Problem 2B.3: $\mathscr{P}_z = p(z) - \rho g z$.

$$\mathscr{P}_z = -\frac{2\mu v_{z,\max}}{B^2} z + \mathscr{P}_0$$

Therefore,

$$\mathscr{P}_0 - \mathscr{P}_z = \frac{2\mu v_{z,\max}}{B^2} z.$$

Problem 2B.6

This problem considers the flow of a film outside a circular tube.

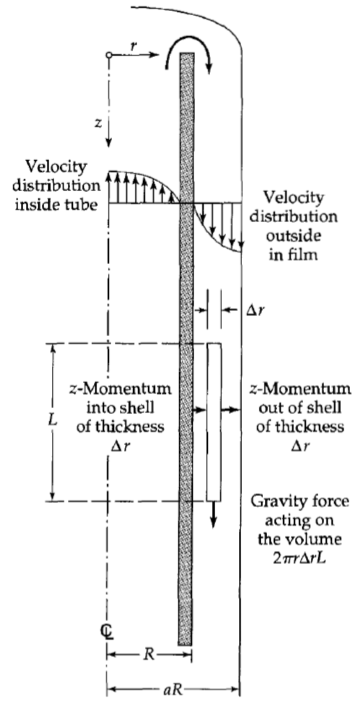

Fig. 2B.6 Velocity distribution and *z*-momentum balance for the flow of a falling film on the outside of a circular tube.

The velocity is assumed to flow only in the z-direction and vary in the r-direction.

$$\mathbf{v} = v_z(r)\hat{\mathbf{z}}$$

Expand the vorticity in cylindrical coordinates by using formulas (G), (H), and (I) on page 834. It only has a θ-component.

$$\mathbf{w} = \nabla \times \mathbf{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)\hat{\mathbf{z}} = -\frac{dv_z}{dr}\hat{\boldsymbol{\theta}}$$

Substitute this into equation (1).

$$\underbrace{\frac{\partial}{\partial t}\left(-\frac{dv_z}{dr}\hat{\boldsymbol{\theta}}\right)}_{=\mathbf{0}} + \mathbf{v}\cdot\nabla\left(-\frac{dv_z}{dr}\hat{\boldsymbol{\theta}}\right) = \nu\nabla^2\left(-\frac{dv_z}{dr}\hat{\boldsymbol{\theta}}\right) + \left(-\frac{dv_z}{dr}\hat{\boldsymbol{\theta}}\right)\cdot\nabla\mathbf{v}$$

The components of $\mathbf{v}\cdot\nabla\mathbf{w}$ and $\mathbf{w}\cdot\nabla\mathbf{v}$ in cylindrical coordinates are given in the formulas at the bottom of page 834. Since only v_z and w_θ are nonzero, only the last term in equation (Q) remains in the former and only the second term in equation (R) remains in the latter.

$$\underbrace{v_z(r)\frac{\partial}{\partial z}\left(-\frac{dv_z}{dr}\right)\hat{\boldsymbol{\theta}}}_{=\mathbf{0}} = \nu\nabla^2\left(-\frac{dv_z}{dr}\hat{\boldsymbol{\theta}}\right) + \underbrace{\left(-\frac{dv_z}{dr}\right)\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta}\right)\hat{\boldsymbol{\theta}}}_{=\mathbf{0}}$$

Because v_z is only a function of r, these terms are zero anyway. Lastly, use equation (N) on page 834 to write out the Laplacian.

$$\mathbf{0} = \nu\left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(-r\frac{dv_z}{dr}\right)\right] + \underbrace{\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\left(-\frac{dv_z}{dr}\right)}_{=0} + \underbrace{\frac{\partial^2}{\partial z^2}\left(-\frac{dv_z}{dr}\right)}_{=0} + \underbrace{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}}_{=0}\right\}\hat{\boldsymbol{\theta}}$$

Dot both sides by $\hat{\boldsymbol{\theta}}$ to get a scalar equation.

$$0 = -\nu \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)\right]$$

Divide both sides by $-\nu$.

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)\right] = 0$$

Integrate both sides with respect to r.

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = C_1$$

Multiply both sides by r.

$$\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = C_1 r$$

Integrate both sides with respect to r once more.

$$r\frac{dv_z}{dr} = \frac{1}{2}C_1 r^2 + C_2$$

Divide both sides by r.

$$\frac{dv_z}{dr} = \frac{1}{2}C_1 r + \frac{C_2}{r}$$

Integrate both sides with respect to r once more.

$$v_z(r) = \frac{1}{4}C_1 r^2 + C_2 \ln r + C_3$$

Three boundary conditions are needed to solve for C_1 and C_2 and C_3. The first comes from the assumption that the fluid does not slip on the tube's surface.

$$v_z(R) = 0$$

The second comes from the fact that because the film is exposed to the air at $r = aR$, the shear stress there is zero.

$$\tau_{rz}(aR) = 0 \quad \Rightarrow \quad \frac{dv_z}{dr}(aR) = 0$$

The third comes from the fact that the fluid flows down the tube due to gravity, not due to a pressure gradient.

$$\frac{\partial p}{\partial z} = 0$$

Apply the first two.

$$v_z(R) = \frac{1}{4}C_1 R^2 + C_2 \ln R + C_3 = 0$$

$$\frac{dv_z}{dr}(aR) = \frac{1}{2}C_1 aR + \frac{C_2}{aR} = 0$$

Solve this system of equations in terms of C_1.

$$C_2 = -\frac{C_1 a^2 R^2}{2}$$

$$C_3 = \frac{C_1 R^2}{4}(2a^2 \ln R - 1)$$

So then

$$\begin{aligned} v_z(r) &= \frac{1}{4}C_1 r^2 + C_2 \ln r + C_3 \\ &= \frac{1}{4}C_1 r^2 - \frac{C_1 a^2 R^2}{2}\ln r + \frac{C_1 R^2}{4}(2a^2 \ln R - 1). \end{aligned}$$

The Navier-Stokes equation is needed now since the pressure is involved.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla \cdot \rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}$$

From Appendix B.6 on page 848, it yields the following three scalar equations in cylindrical coordinates.

$$\rho\Bigg(\underbrace{\frac{\partial v_r}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_r}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_r}{\partial z}}_{=0} - \underbrace{\frac{v_\theta^2}{r}}_{=0}\Bigg) = -\frac{\partial p}{\partial r} + \mu\Bigg[\underbrace{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)}_{=0} + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_r}{\partial z^2}}_{=0} - \underbrace{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}}_{=0}\Bigg] + \rho g_r$$

$$\rho\Bigg(\underbrace{\frac{\partial v_\theta}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_\theta}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_\theta}{\partial z}}_{=0} + \underbrace{\frac{v_r v_\theta}{r}}_{=0}\Bigg) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\Bigg[\underbrace{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right)}_{=0} + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_\theta}{\partial z^2}}_{=0} + \underbrace{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}}_{=0}\Bigg] + \rho g_\theta$$

$$\rho\Bigg(\underbrace{\frac{\partial v_z}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_z}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_z}{\partial z}}_{=0}\Bigg) = \underbrace{-\frac{\partial p}{\partial z}}_{=0} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_z}{\partial z^2}}_{=0}\right] + \rho g_z$$

Gravity points down in the positive z-direction, so $\mathbf{g} = g\hat{\mathbf{z}}$, which means $g_r = 0$, $g_\theta = 0$, and $g_z = g$. The z-equation is the relevant one.

$$\frac{\mu}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) + \rho g = 0$$

$$\mu C_1 + \rho g = 0$$

$$C_1 = -\frac{\rho g}{\mu}$$

Therefore,

$$\begin{aligned} v_z(r) &= \frac{1}{4}C_1 r^2 - \frac{C_1 a^2 R^2}{2}\ln r + \frac{C_1 R^2}{4}(2a^2 \ln R - 1) \\ &= -\frac{1}{4}\frac{\rho g}{\mu}r^2 + \frac{\rho g a^2 R^2}{2\mu}\ln r - \frac{\rho g R^2}{4\mu}(2a^2 \ln R - 1) \\ &= \frac{\rho g R^2}{4\mu}\left(-\frac{r^2}{R^2} + 2a^2 \ln r - 2a^2 \ln R + 1\right) \\ &= \frac{\rho g R^2}{4\mu}\left[1 - \left(\frac{r}{R}\right)^2 + 2a^2 \ln\left(\frac{r}{R}\right)\right]. \end{aligned}$$

Problem 2B.7

This problem considers the flow in an annulus with the inner cylinder moving axially.

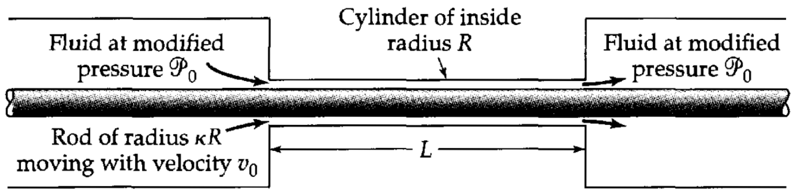

Fig. 2B.7 Annular flow with the inner cylinder moving axially.

The velocity is assumed to flow only in the z-direction and vary in the r-direction.

$$\mathbf{v} = v_z(r)\hat{\mathbf{z}}$$

Expand the vorticity in cylindrical coordinates by using formulas (G), (H), and (I) on page 834. It only has a θ-component.

$$\mathbf{w} = \nabla \times \mathbf{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)\hat{\mathbf{z}} = -\frac{dv_z}{dr}\hat{\boldsymbol{\theta}}$$

Substitute this into equation (1).

$$\underbrace{\frac{\partial}{\partial t}\left(-\frac{dv_z}{dr}\hat{\boldsymbol{\theta}}\right)}_{=\mathbf{0}} + \mathbf{v}\cdot\nabla\left(-\frac{dv_z}{dr}\hat{\boldsymbol{\theta}}\right) = \nu\nabla^2\left(-\frac{dv_z}{dr}\hat{\boldsymbol{\theta}}\right) + \left(-\frac{dv_z}{dr}\hat{\boldsymbol{\theta}}\right)\cdot\nabla\mathbf{v}$$

The components of $\mathbf{v}\cdot\nabla\mathbf{w}$ and $\mathbf{w}\cdot\nabla\mathbf{v}$ in cylindrical coordinates are given in the formulas at the bottom of page 834. Since only v_z and w_θ are nonzero, only the last term in equation (Q) remains in the former and only the second term in equation (R) remains in the latter.

$$\underbrace{v_z(r)\frac{\partial}{\partial z}\left(-\frac{dv_z}{dr}\right)\hat{\boldsymbol{\theta}}}_{=\mathbf{0}} = \nu\nabla^2\left(-\frac{dv_z}{dr}\hat{\boldsymbol{\theta}}\right) + \underbrace{\left(-\frac{dv_z}{dr}\right)\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta}\right)\hat{\boldsymbol{\theta}}}_{=\mathbf{0}}$$

Because v_z is only a function of r, these terms are zero anyway. Lastly, use equation (N) on page 834 to write out the Laplacian.

$$\mathbf{0} = \nu\left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(-r\frac{dv_z}{dr}\right)\right] + \underbrace{\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\left(-\frac{dv_z}{dr}\right)}_{=0} + \underbrace{\frac{\partial^2}{\partial z^2}\left(-\frac{dv_z}{dr}\right)}_{=0} + \underbrace{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}}_{=0}\right\}\hat{\boldsymbol{\theta}}$$

Dot both sides by $\hat{\boldsymbol{\theta}}$ to get a scalar equation.

$$0 = -\nu\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)\right]$$

Divide both sides by $-\nu$.

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)\right] = 0$$

Integrate both sides with respect to r.

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = C_1$$

Multiply both sides by r.

$$\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = C_1 r$$

Integrate both sides with respect to r once more.

$$r\frac{dv_z}{dr} = \frac{1}{2}C_1 r^2 + C_2$$

Divide both sides by r.

$$\frac{dv_z}{dr} = \frac{1}{2}C_1 r + \frac{C_2}{r}$$

Integrate both sides with respect to r once more.

$$v_z(r) = \frac{1}{4}C_1 r^2 + C_2 \ln r + C_3$$

Three boundary conditions are needed to solve for C_1 and C_2 and C_3. Two of them come from the assumption that the fluid does not slip on the walls.

$$v_z(\kappa R) = v_0$$
$$v_z(R) = 0$$

The third comes from the fact that the fluid flows due to the cylinder's motion, not due to a pressure gradient or gravity.

$$\frac{\partial p}{\partial z} = 0 \qquad g_z = 0$$

The Navier-Stokes equation is needed since the pressure and gravity are involved.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla\cdot\rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}$$

From Appendix B.6 on page 848, it yields the following three scalar equations in cylindrical coordinates.

$$\rho\Bigg(\underbrace{\frac{\partial v_r}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_r}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_r}{\partial z}}_{=0} - \underbrace{\frac{v_\theta^2}{r}}_{=0}\Bigg) = -\frac{\partial p}{\partial r} + \mu\Bigg[\underbrace{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)}_{=0} + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_r}{\partial z^2}}_{=0} - \underbrace{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}}_{=0}\Bigg] + \rho g_r$$

$$\rho\Bigg(\underbrace{\frac{\partial v_\theta}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_\theta}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_\theta}{\partial z}}_{=0} + \underbrace{\frac{v_r v_\theta}{r}}_{=0}\Bigg) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\Bigg[\underbrace{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right)}_{=0} + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_\theta}{\partial z^2}}_{=0} + \underbrace{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}}_{=0}\Bigg] + \rho g_\theta$$

$$\rho\Bigg(\underbrace{\frac{\partial v_z}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_z}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_z}{\partial z}}_{=0}\Bigg) = -\underbrace{\frac{\partial p}{\partial z}}_{=0} + \mu\Bigg[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_z}{\partial z^2}}_{=0}\Bigg] + \underbrace{\rho g_z}_{=0}$$

The z-equation is the relevant one.

$$\begin{aligned} 0 &= \frac{\mu}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) \\ &= \mu C_1 \end{aligned}$$

Divide both sides by μ.

$$C_1 = 0$$

Now apply the first two boundary conditions to determine C_2 and C_3.

$$v_z(\kappa R) = \frac{1}{4}C_1\kappa^2 R^2 + C_2 \ln \kappa R + C_3 = v_0$$
$$v_z(R) = \frac{1}{4}C_1 R^2 + C_2 \ln R + C_3 = 0$$

Solve this system of equations.

$$C_2 = \frac{v_0}{\ln \kappa}$$
$$C_3 = -\frac{v_0 \ln R}{\ln \kappa}$$

So then

$$\begin{aligned} v_z(r) &= \frac{1}{4}C_1 r^2 + C_2 \ln r + C_3 \\ &= \frac{v_0}{\ln \kappa}\ln r - \frac{v_0 \ln R}{\ln \kappa} \\ &= v_0 \frac{\ln r - \ln R}{\ln \kappa} \\ &= v_0 \frac{\ln(r/R)}{\ln \kappa}. \end{aligned}$$

Therefore, dividing both sides by v_0,

$$\frac{v_z}{v_0} = \frac{\ln(r/R)}{\ln \kappa}.$$

Problem 3B.1

This problem considers the flow of a fluid between two concentric cylinders. The inner cylinder has radius κR and rotates with angular velocity $\boldsymbol{\Omega} = \Omega_i \hat{\mathbf{z}}$, and the outer cylinder has radius R and rotates with angular velocity $\boldsymbol{\Omega} = \Omega_o \hat{\mathbf{z}}$. It's assumed that the fluid flows only in the θ-direction and that its tangential velocity varies with radius.

$$\mathbf{v} = v_\theta(r)\hat{\boldsymbol{\theta}}$$

Expand the vorticity in cylindrical coordinates by using formulas (G), (H), and (I) on page 834. It only has a z-component.

$$\mathbf{w} = \nabla \times \mathbf{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)\hat{\mathbf{z}} = \frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\hat{\mathbf{z}}$$

Substitute this into equation (1).

$$\underbrace{\frac{\partial}{\partial t}\left[\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\hat{\mathbf{z}}\right]}_{=\mathbf{0}} + \mathbf{v}\cdot\nabla\left[\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\hat{\mathbf{z}}\right] = \nu\nabla^2\left[\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\hat{\mathbf{z}}\right] + \left[\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\hat{\mathbf{z}}\right]\cdot\nabla\mathbf{v}$$

The components of $\mathbf{v}\cdot\nabla\mathbf{w}$ and $\mathbf{w}\cdot\nabla\mathbf{v}$ in cylindrical coordinates are given in the formulas at the bottom of page 834. Since only v_θ and w_z are nonzero, only the second term in equation (R) remains in the former and only the last term in equation (Q) remains in the latter.

$$\underbrace{\frac{v_\theta}{r}\frac{\partial}{\partial \theta}\left[\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right]\hat{\mathbf{z}}}_{=\mathbf{0}} = \nu\nabla^2\left[\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\hat{\mathbf{z}}\right] + \underbrace{\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\frac{\partial v_\theta}{\partial z}\hat{\boldsymbol{\theta}}}_{=\mathbf{0}}$$

Because v_θ is only a function of r, these terms are zero anyway. Lastly, use equation (O) on page 834 to write out the Laplacian.

$$\mathbf{0} = \nu\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right)\right] + \underbrace{\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right)}_{=0} + \underbrace{\frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right)}_{=0}\right\}\hat{\mathbf{z}}$$

Dot both sides by $\hat{\mathbf{z}}$ to get a scalar equation.

$$0 = \frac{\nu}{r}\frac{d}{dr}\left[r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rv_\theta)\right)\right]$$

Multiply both sides by r/ν.

$$\frac{d}{dr}\left[r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rv_\theta)\right)\right] = 0$$

Integrate both sides with respect to r.

$$r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rv_\theta)\right) = C_1$$

Divide both sides by r.

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rv_\theta)\right) = \frac{C_1}{r}$$

Integrate both sides with respect to r once more.

$$\frac{1}{r}\frac{d}{dr}(rv_\theta) = C_1 \ln r + C_2$$

Multiply both sides by r.

$$\frac{d}{dr}(rv_\theta) = C_1 r \ln r + C_2 r$$

Integrate both sides with respect to r once more.

$$rv_\theta = C_1 \frac{r^2}{4}(2\ln r - 1) + C_2\frac{r^2}{2} + C_3$$

Divide both sides by r.

$$v_\theta(r) = C_1 \frac{r}{4}(2\ln r - 1) + C_2\frac{r}{2} + \frac{C_3}{r}$$

Three boundary conditions are needed to solve for C_1 and C_2 and C_3. Two of them come from the assumption that the fluid does not slip on the walls. Multiply the cylinders' angular velocities by their respective moment arms to get the tangential velocities.

$$v_\theta(\kappa R) = \kappa R\Omega_i$$
$$v_\theta(R) = R\Omega_o$$

The third comes from the fact that the fluid flows due to the cylinders' motion, not due to a pressure gradient or gravity.

$$\frac{\partial p}{\partial \theta} = 0 \qquad g_\theta = 0$$

The Navier-Stokes equation is needed since the pressure and gravity are involved.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla\cdot\rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}$$

From Appendix B.6 on page 848, it yields the following three scalar equations in cylindrical coordinates.

$$\rho\left(\underbrace{\frac{\partial v_r}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_r}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_r}{\partial z}}_{=0} - \frac{v_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left[\underbrace{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)}_{=0} + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_r}{\partial z^2}}_{=0} - \underbrace{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}}_{=0}\right] + \rho g_r$$

$$\rho\left(\underbrace{\frac{\partial v_\theta}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_\theta}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_\theta}{\partial z}}_{=0} + \underbrace{\frac{v_r v_\theta}{r}}_{=0}\right) = \underbrace{-\frac{1}{r}\frac{\partial p}{\partial \theta}}_{=0} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_\theta}{\partial z^2}}_{=0} + \underbrace{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}}_{=0}\right] + \underbrace{\rho g_\theta}_{=0}$$

$$\rho\left(\underbrace{\frac{\partial v_z}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_z}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_z}{\partial z}}_{=0}\right) = -\frac{\partial p}{\partial z} + \mu\left[\underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)}_{=0} + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_z}{\partial z^2}}_{=0}\right] + \rho g_z$$

Consider the θ-equation only.

$$\begin{aligned} 0 &= \mu\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rv_\theta)\right) \\ &= \mu\frac{C_1}{r} \end{aligned}$$

Divide both sides by μ/r.

$$C_1 = 0$$

Now apply the first two boundary conditions to determine C_2 and C_3.

$$\begin{aligned} v_\theta(\kappa R) &= C_2 \frac{\kappa R}{2} + \frac{C_3}{\kappa R} = \kappa R \Omega_i \\ v_\theta(R) &= C_2 \frac{R}{2} + \frac{C_3}{R} = R\Omega_o \end{aligned}$$

Solving this system of equations yields

$$C_2 = \frac{2(\Omega_o - \kappa^2\Omega_i)}{1-\kappa^2} \quad \text{and} \quad C_3 = \frac{\kappa^2 R^2 (\Omega_i - \Omega_o)}{1-\kappa^2}.$$

Therefore,

$$\begin{aligned} v_\theta(r) &= C_1 \frac{r}{4}(2\ln r - 1) + C_2 \frac{r}{2} + \frac{C_3}{r} \\ &= \frac{2(\Omega_o - \kappa^2\Omega_i)}{1-\kappa^2}\frac{r}{2} + \frac{\kappa^2 R^2(\Omega_i - \Omega_o)}{1-\kappa^2}\frac{1}{r} \\ &= \frac{\kappa R}{1-\kappa^2}\left[(\Omega_o - \kappa^2\Omega_i)\frac{r}{\kappa R} + (\Omega_i - \Omega_o)\frac{\kappa R}{r}\right]. \end{aligned}$$

For part (b), the velocity between two concentric rotating spheres is in the azimuthal ϕ-direction and varies as a function of r and θ (polar angle).

$$\mathbf{v} = v_\phi(r,\theta)\hat{\boldsymbol{\phi}}$$

This results in a vorticity with two components, one in the $\hat{\mathbf{r}}$-direction and one in the $\hat{\boldsymbol{\theta}}$-direction, unlike the other problems which only have one.

$$\begin{aligned} \mathbf{w} = \nabla \times \mathbf{v} &= \left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\phi \sin\theta) - \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \left[\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \left[\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}} \\ &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\phi\sin\theta)\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial}{\partial r}(r v_\phi)\hat{\boldsymbol{\theta}} \end{aligned}$$

Part (b) will be skipped.

Problem 3B.6

This problem considers circulating axial flow in an annulus.

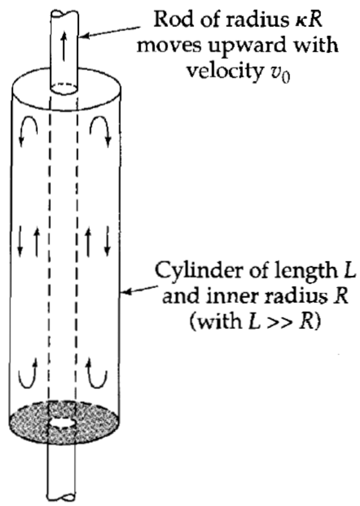

Fig. 3B.6. Circulating flow produced by an axially moving rod in a closed annular region.

The velocity is assumed to flow only in the z-direction and vary in the r-direction.

$$\mathbf{v} = v_z(r)\hat{\mathbf{z}}$$

Expand the vorticity in cylindrical coordinates by using formulas (G), (H), and (I) on page 834. It only has a θ-component.

$$\mathbf{w} = \nabla \times \mathbf{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)\hat{\mathbf{z}} = -\frac{dv_z}{dr}\hat{\boldsymbol{\theta}}$$

Substitute this into equation (1).

$$\underbrace{\frac{\partial}{\partial t}\left(-\frac{dv_z}{dr}\hat{\boldsymbol{\theta}}\right)}_{=\mathbf{0}} + \mathbf{v}\cdot\nabla\left(-\frac{dv_z}{dr}\hat{\boldsymbol{\theta}}\right) = \nu\nabla^2\left(-\frac{dv_z}{dr}\hat{\boldsymbol{\theta}}\right) + \left(-\frac{dv_z}{dr}\hat{\boldsymbol{\theta}}\right)\cdot\nabla\mathbf{v}$$

The components of $\mathbf{v}\cdot\nabla\mathbf{w}$ and $\mathbf{w}\cdot\nabla\mathbf{v}$ in cylindrical coordinates are given in the formulas at the bottom of page 834. Since only v_z and w_θ are nonzero, only the last term in equation (Q) remains in the former and only the second term in equation (R) remains in the latter.

$$\underbrace{v_z(r)\frac{\partial}{\partial z}\left(-\frac{dv_z}{dr}\right)\hat{\boldsymbol{\theta}}}_{=\mathbf{0}} = \nu\nabla^2\left(-\frac{dv_z}{dr}\hat{\boldsymbol{\theta}}\right) + \underbrace{\left(-\frac{dv_z}{dr}\right)\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta}\right)\hat{\boldsymbol{\theta}}}_{=\mathbf{0}}$$

Because v_z is only a function of r, these terms are zero anyway. Lastly, use equation (N) on page 834 to write out the Laplacian.

$$\mathbf{0} = \nu\left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(-r\frac{dv_z}{dr}\right)\right] + \underbrace{\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\left(-\frac{dv_z}{dr}\right)}_{=0} + \underbrace{\frac{\partial^2}{\partial z^2}\left(-\frac{dv_z}{dr}\right)}_{=0} + \underbrace{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}}_{=0}\right\}\hat{\boldsymbol{\theta}}$$

Dot both sides by $\hat{\boldsymbol{\theta}}$ to get a scalar equation.

$$0 = -\nu \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)\right]$$

Divide both sides by $-\nu$.

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)\right] = 0$$

Integrate both sides with respect to r.

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = C_1$$

Multiply both sides by r.

$$\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = C_1 r$$

Integrate both sides with respect to r once more.

$$r\frac{dv_z}{dr} = \frac{1}{2}C_1 r^2 + C_2$$

Divide both sides by r.

$$\frac{dv_z}{dr} = \frac{1}{2}C_1 r + \frac{C_2}{r}$$

Integrate both sides with respect to r once more.

$$v_z(r) = \frac{1}{4}C_1 r^2 + C_2 \ln r + C_3$$

Because two boundary conditions are provided, two of the three constants can be determined. Apply them now.

$$v_z(R) = \frac{1}{4}C_1 R^2 + C_2 \ln R + C_3 = 0 \tag{2}$$

$$v_z(\kappa R) = \frac{1}{4}C_1 \kappa^2 R^2 + C_2 \ln \kappa R + C_3 = v_0 \tag{3}$$

The third equation comes from the fact that the fluid circulates in the closed annular region; that is, through any given cross-section, the volumetric flow rate must be zero.

$$\int \mathbf{v} \cdot d\mathbf{A} = 0$$

$$\int v_z(r)\, dA = 0$$

$$\int_{\kappa R}^{R} \left(\frac{1}{4}C_1 r^2 + C_2 \ln r + C_3\right)(2\pi r\, dr) = 0$$

$$2\pi\left(\int_{\kappa R}^{R} \frac{1}{4}C_1 r^3\, dr + \int_{\kappa R}^{R} C_2 r \ln r\, dr + \int_{\kappa R}^{R} C_3 r\, dr\right) = 0$$

$$2\pi\left[\frac{1}{16}C_1 R^4(1-\kappa^4) + \frac{1}{4}C_2 R^2(\kappa^2 - 1 + 2\ln R - 2\kappa^2 \ln \kappa R) + \frac{1}{2}C_3 R^2(1-\kappa^2)\right] = 0 \tag{4}$$

Solve equations (2), (3), and (4) for C_1, C_2, and C_3.

$$C_1 = \frac{4v_0\left(1-\kappa^2-2\kappa^2\ln\frac{1}{\kappa}\right)}{R^2\left[(1-\kappa^4)\ln\frac{1}{\kappa}-(1-\kappa^2)^2\right]}$$

$$C_2 = \frac{v_0(1-\kappa^2)}{1-\kappa^2-(1+\kappa^2)\ln\frac{1}{\kappa}}$$

$$C_3 = \frac{v_0[1-\kappa^2-(1+\kappa^4)\ln R+2\kappa^2\ln\kappa R]}{(1-\kappa^2)^2-(1-\kappa^4)\ln\frac{1}{\kappa}}$$

So then

$$\begin{aligned}
v_z(r) &= \frac{1}{4}C_1 r^2 + C_2\ln r + C_3 \\
&= \frac{1}{4}\frac{4v_0\left(1-\kappa^2-2\kappa^2\ln\frac{1}{\kappa}\right)}{R^2\left[(1-\kappa^4)\ln\frac{1}{\kappa}-(1-\kappa^2)^2\right]}r^2 + \frac{v_0(1-\kappa^2)}{1-\kappa^2-(1+\kappa^2)\ln\frac{1}{\kappa}}\ln r + \frac{v_0(1-\kappa^2-(1+\kappa^4)\ln R+2\kappa^2\ln\kappa R)}{(1-\kappa^2)^2-(1-\kappa^4)\ln\frac{1}{\kappa}} \\
&= \frac{v_0\left(1-\kappa^2-2\kappa^2\ln\frac{1}{\kappa}\right)}{(1-\kappa^4)\ln\frac{1}{\kappa}-(1-\kappa^2)^2}\left(\frac{r}{R}\right)^2 + \frac{v_0(1-\kappa^2)}{1-\kappa^2-(1+\kappa^2)\ln\frac{1}{\kappa}}\left(\ln\frac{r}{R}+\ln R\right) - \frac{v_0[1-\kappa^2-(1+\kappa^4)\ln R+2\kappa^2\ln\kappa R]}{(1-\kappa^4)\ln\frac{1}{\kappa}-(1-\kappa^2)^2}.
\end{aligned}$$

Therefore, dividing both sides by v_0 and letting $\xi = r/R$,

$$\begin{aligned}
\frac{v_z}{v_0} &= \frac{\left(1-\kappa^2-2\kappa^2\ln\frac{1}{\kappa}\right)\xi^2-[1-\kappa^2-(1+\kappa^4)\ln R+2\kappa^2\ln\kappa R]}{(1-\kappa^4)\ln\frac{1}{\kappa}-(1-\kappa^2)^2} + \frac{1-\kappa^2}{1-\kappa^2-(1+\kappa^2)\ln\frac{1}{\kappa}}\ln\xi + \frac{1-\kappa^2}{1-\kappa^2-(1+\kappa^2)\ln\frac{1}{\kappa}}\ln R \\
&= \frac{\left(1-\kappa^2-2\kappa^2\ln\frac{1}{\kappa}\right)\xi^2-[1-\kappa^2-(1+\kappa^4)\ln R+2\kappa^2(\ln\kappa+\ln R)]}{-(1-\kappa^2)\left[(1-\kappa^2)-(1+\kappa^2)\ln\frac{1}{\kappa}\right]} - \frac{1-\kappa^2}{(1-\kappa^2)-(1+\kappa^2)\ln\frac{1}{\kappa}}\ln\frac{1}{\xi} + \frac{1-\kappa^2}{(1-\kappa^2)-(1+\kappa^2)\ln\frac{1}{\kappa}}\ln R \\
&= \frac{[1-\kappa^2-(1-2\kappa^2+\kappa^4)\ln R+2\kappa^2\ln\kappa]-\left(1-\kappa^2-2\kappa^2\ln\frac{1}{\kappa}\right)\xi^2}{(1-\kappa^2)\left[(1-\kappa^2)-(1+\kappa^2)\ln\frac{1}{\kappa}\right]} - \frac{1-\kappa^2}{(1-\kappa^2)-(1+\kappa^2)\ln\frac{1}{\kappa}}\ln\frac{1}{\xi} + \frac{1-\kappa^2}{(1-\kappa^2)-(1+\kappa^2)\ln\frac{1}{\kappa}}\ln R \\
&= \frac{\left[1-(1-\kappa^2)\ln R+\frac{2\kappa^2}{1-\kappa^2}\ln\kappa\right]-\left(1-\frac{2\kappa^2}{1-\kappa^2}\ln\frac{1}{\kappa}\right)\xi^2}{(1-\kappa^2)-(1+\kappa^2)\ln\frac{1}{\kappa}} - \frac{(1-\kappa^2)\ln\frac{1}{\xi}}{(1-\kappa^2)-(1+\kappa^2)\ln\frac{1}{\kappa}} + \frac{(1-\kappa^2)\ln R}{(1-\kappa^2)-(1+\kappa^2)\ln\frac{1}{\kappa}} \\
&= \frac{\left[1+\frac{2\kappa^2}{1-\kappa^2}\ln\kappa\right]-\left(1-\frac{2\kappa^2}{1-\kappa^2}\ln\frac{1}{\kappa}\right)\xi^2}{(1-\kappa^2)-(1+\kappa^2)\ln\frac{1}{\kappa}} - \frac{(1-\kappa^2)\ln\frac{1}{\xi}}{(1-\kappa^2)-(1+\kappa^2)\ln\frac{1}{\kappa}} \\
&= \frac{(1-\xi^2)\left(1-\frac{2\kappa^2}{1-\kappa^2}\ln\frac{1}{\kappa}\right)-(1-\kappa^2)\ln\frac{1}{\xi}}{(1-\kappa^2)-(1+\kappa^2)\ln\frac{1}{\kappa}}.
\end{aligned}$$

To get the pressure distribution, the Navier-Stokes equation is needed.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla \cdot \rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}$$

From Appendix B.6 on page 848, it yields the following three scalar equations in cylindrical coordinates.

$$\rho\left(\underbrace{\frac{\partial v_r}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_r}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_r}{\partial z}}_{=0} - \underbrace{\frac{v_\theta^2}{r}}_{=0}\right) = -\frac{\partial p}{\partial r} + \mu\left[\underbrace{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)}_{=0} + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_r}{\partial z^2}}_{=0} - \underbrace{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}}_{=0}\right] + \rho g_r$$

$$\rho\left(\underbrace{\frac{\partial v_\theta}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_\theta}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_\theta}{\partial z}}_{=0} + \underbrace{\frac{v_r v_\theta}{r}}_{=0}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\underbrace{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right)}_{=0} + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_\theta}{\partial z^2}}_{=0} + \underbrace{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}}_{=0}\right] + \rho g_\theta$$

$$\rho\left(\underbrace{\frac{\partial v_z}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_z}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_z}{\partial z}}_{=0}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_z}{\partial z^2}}_{=0}\right] + \rho g_z$$

Gravity points down in the negative z-direction, so $\mathbf{g} = -g\hat{\mathbf{z}}$, which means $g_r = 0$, $g_\theta = 0$, and $g_z = -g$.

$$\begin{aligned}
0 &= -\frac{\partial p}{\partial r} \\
0 &= -\frac{1}{r}\frac{\partial p}{\partial \theta} \\
0 &= -\frac{\partial p}{\partial z} + \frac{\mu}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) - \rho g
\end{aligned}$$

These first two equations imply that p is neither a function of r nor a function of θ: $p = p(z)$.

$$\begin{aligned}
0 &= -\frac{dp}{dz} + \frac{\mu}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) - \rho g \\
&= -\frac{d}{dz}(p + \rho g z) + \frac{\mu}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)
\end{aligned}$$

Define the modified pressure here to be $\mathscr{P} = p(z) + \rho g z$.

$$0 = -\frac{d\mathscr{P}}{dz} + \frac{\mu}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)$$

Solve for it.

$$\begin{aligned}
\frac{d\mathscr{P}}{dz} &= \frac{\mu}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) \\
&= \mu C_1 \\
&= \frac{4\mu v_0\left(1 - \kappa^2 - 2\kappa^2 \ln\frac{1}{\kappa}\right)}{R^2\left[(1 - \kappa^4)\ln\frac{1}{\kappa} - (1 - \kappa^2)^2\right]}
\end{aligned}$$

Integrate both sides with respect to z.

$$\mathscr{P} = \frac{4\mu v_0\left(1 - \kappa^2 - 2\kappa^2 \ln\frac{1}{\kappa}\right)}{R^2\left[(1 - \kappa^4)\ln\frac{1}{\kappa} - (1 - \kappa^2)^2\right]}z + \mathscr{P}_0$$

Problem 3B.10

This problem considers the radial flow between two parallel disks.

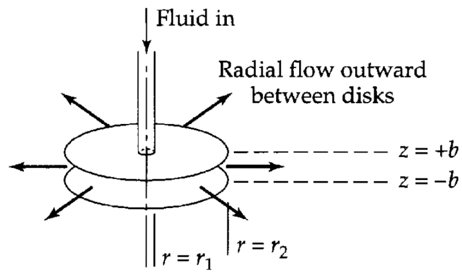

Fig. 3B.10. Outward radial flow in the space between two parallel, circular disks.

The velocity is assumed to flow only in the r-direction and vary both radially and vertically.

$$\mathbf{v} = v_r(r, z)\hat{\mathbf{r}}$$

Expand the vorticity in cylindrical coordinates by using formulas (G), (H), and (I) on page 834. It only has a θ-component.

$$\mathbf{w} = \nabla \times \mathbf{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)\hat{\mathbf{z}} = \frac{\partial v_r}{\partial z}\hat{\boldsymbol{\theta}}$$

Substitute this into equation (1).

$$\underbrace{\frac{\partial}{\partial t}\left(\frac{\partial v_r}{\partial z}\hat{\boldsymbol{\theta}}\right)}_{=\,\mathbf{0}} + \mathbf{v}\cdot\nabla\left(\frac{\partial v_r}{\partial z}\hat{\boldsymbol{\theta}}\right) = \nu\nabla^2\left(\frac{\partial v_r}{\partial z}\hat{\boldsymbol{\theta}}\right) + \left(\frac{\partial v_r}{\partial z}\hat{\boldsymbol{\theta}}\right)\cdot\nabla\mathbf{v}$$

The components of $\mathbf{v}\cdot\nabla\mathbf{w}$ and $\mathbf{w}\cdot\nabla\mathbf{v}$ in cylindrical coordinates are given in the formulas at the bottom of page 834. Since only v_r and w_θ are nonzero, only the first term in equation (Q) remains in the former and only the second term in equation (Q) remains in the latter.

$$v_r(r, z)\frac{\partial}{\partial r}\left(\frac{\partial v_r}{\partial z}\right)\hat{\boldsymbol{\theta}} = \nu\nabla^2\left(\frac{\partial v_r}{\partial z}\hat{\boldsymbol{\theta}}\right) + \frac{\partial v_r}{\partial z}\frac{v_r(r, z)}{r}\hat{\boldsymbol{\theta}}$$

Lastly, use equation (N) on page 834 to write out the Laplacian.

$$v_r(r, z)\frac{\partial}{\partial r}\left(\frac{\partial v_r}{\partial z}\right)\hat{\boldsymbol{\theta}} = \nu\left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial z}\right)\right] + \frac{\partial^2}{\partial z^2}\left(\frac{\partial v_r}{\partial z}\right)\right\}\hat{\boldsymbol{\theta}} + \frac{\partial v_r}{\partial z}\frac{v_r(r, z)}{r}\hat{\boldsymbol{\theta}}$$

Dot both sides by $\hat{\boldsymbol{\theta}}$ to get a scalar equation.

$$v_r(r, z)\frac{\partial}{\partial r}\left(\frac{\partial v_r}{\partial z}\right) = \nu\left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial z}\right)\right] + \frac{\partial^2}{\partial z^2}\left(\frac{\partial v_r}{\partial z}\right)\right\} + \frac{\partial v_r}{\partial z}\frac{v_r(r, z)}{r}$$

Invoke the creeping flow assumption in order to neglect the nonlinear terms.

$$0 = \nu\left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial z}\right)\right] + \frac{\partial^2}{\partial z^2}\left(\frac{\partial v_r}{\partial z}\right)\right\}$$

Bring the z-derivative outside the square brackets.

$$0 = \nu \left\{ \frac{\partial^2}{\partial r \partial z} \left[\frac{1}{r} \frac{\partial}{\partial r}(r v_r) \right] + \frac{\partial^2}{\partial z^2} \left(\frac{\partial v_r}{\partial z} \right) \right\}$$

Divide both sides by ν.

$$0 = \frac{\partial^2}{\partial r \partial z} \left[\frac{1}{r} \frac{\partial}{\partial r}(r v_r) \right] + \frac{\partial^3 v_r}{\partial z^3} \tag{5}$$

Since the density of the fluid is constant, the continuity equation, which results from making a mass balance over some volume element that the fluid is flowing through, reduces to

$$\nabla \cdot \mathbf{v} = 0.$$

Use equation (A) on page 834 to expand the left side in cylindrical coordinates.

$$\frac{1}{r} \frac{\partial}{\partial r}(r v_r) + \underbrace{\frac{1}{r} \frac{\partial v_\theta}{\partial \theta}}_{=0} + \underbrace{\frac{\partial v_z}{\partial z}}_{=0} = 0$$

Multiply both sides by r.

$$\frac{\partial}{\partial r}(r v_r) = 0 \tag{6}$$

As a result, the first term in equation (5) vanishes.

$$\frac{\partial^3 v_r}{\partial z^3} = 0 \tag{7}$$

Integrate both sides of equation (6) partially with respect to r.

$$r v_r = f(z)$$

Divide both sides by r.

$$v_r(r, z) = \frac{f(z)}{r}$$

Now substitute this result into equation (7) to determine the arbitrary function $f(z)$.

$$\frac{\partial^3}{\partial z^3} \left[\frac{f(z)}{r} \right] = 0$$

$$\frac{1}{r} \frac{d^3 f}{dz^3} = 0$$

Multiply both sides by r.

$$\frac{d^3 f}{dz^3} = 0$$

Integrate both sides with respect to z.

$$\frac{d^2 f}{dz^2} = C_1$$

Integrate both sides with respect to z once more.

$$\frac{df}{dz} = C_1 z + C_2$$

Integrate both sides with respect to z once more.

$$f(z) = \frac{C_1}{2}z^2 + C_2 z + C_3$$

Consequently, the radial velocity is

$$v_r(r,z) = \frac{1}{r}\left(\frac{C_1}{2}z^2 + C_2 z + C_3\right).$$

Three boundary conditions are needed to determine C_1, C_2, and C_3. Two of them are obtained from the assumption that the fluid does not slip on the surfaces at $z = \pm b$.

$$v_r(r,-b) = 0$$
$$v_r(r,b) = 0$$

The third is from the fact that the fluid is flowing due to an imposed modified pressure gradient in the radial direction: At $r = r_1$ the modified pressure is $\mathscr{P} = \mathscr{P}_1$, and at $r = r_2$ the modified pressure is $\mathscr{P} = \mathscr{P}_2$. Apply the first two boundary conditions.

$$v_r(r,-b) = \frac{1}{r}\left(\frac{C_1}{2}b^2 - C_2 b + C_3\right) = 0$$
$$v_r(r,b) = \frac{1}{r}\left(\frac{C_1}{2}b^2 + C_2 b + C_3\right) = 0$$

Solving this system of equations for C_1 and C_2 yields

$$C_1 = -\frac{2C_3}{b^2} \quad \text{and} \quad C_2 = 0,$$

which means the radial velocity becomes

$$\begin{aligned} v_r(r,z) &= \frac{1}{r}\left(-\frac{C_3}{b^2}z^2 + C_3\right) \\ &= \frac{C_3}{r}\left(1 - \frac{z^2}{b^2}\right) \\ &= \frac{C_3}{r}\left[1 - \left(\frac{z}{b}\right)^2\right]. \end{aligned}$$

Since the modified pressure is involved, it's necessary to consider the Navier-Stokes equation.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla \cdot \rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}$$

From Appendix B.6 on page 848, it yields the following three scalar equations in cylindrical coordinates. (The creeping flow assumption was made, so all nonlinear terms on each left side are neglected.)

$$\rho\Bigg(\underbrace{\frac{\partial v_r}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_r}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_r}{\partial z}}_{=0} - \underbrace{\frac{v_\theta^2}{r}}_{=0}\Bigg) = -\frac{\partial p}{\partial r} + \mu\Bigg[\underbrace{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)}_{=0} + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}}_{=0} + \frac{\partial^2 v_r}{\partial z^2} - \underbrace{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}}_{=0}\Bigg] + \rho g_r$$

$$\rho\Bigg(\underbrace{\frac{\partial v_\theta}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_\theta}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_\theta}{\partial z}}_{=0} + \underbrace{\frac{v_r v_\theta}{r}}_{=0}\Bigg) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\Bigg[\underbrace{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right)}_{=0} + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_\theta}{\partial z^2}}_{=0} + \underbrace{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}}_{=0}\Bigg] + \rho g_\theta$$

$$\rho\Bigg(\underbrace{\frac{\partial v_z}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_z}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_z}{\partial z}}_{=0}\Bigg) = -\frac{\partial p}{\partial z} + \mu\Bigg[\underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)}_{=0} + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_z}{\partial z^2}}_{=0}\Bigg] + \rho g_z$$

Gravity points down in the negative z-direction, so $\mathbf{g} = -g\hat{\mathbf{z}}$, which means $g_r = 0$, $g_\theta = 0$, and $g_z = -g$. The r- and z-equations are relevant here.

$$0 = -\frac{\partial p}{\partial r} + \mu \frac{\partial^2 v_r}{\partial z^2} \tag{8}$$

$$0 = -\frac{\partial p}{\partial z} - \rho g \tag{9}$$

Combine the pressure and gravity terms in equation (9) by introducing a modified pressure function.

$$0 = -\frac{\partial}{\partial z}(p + \rho g z)$$

Let $\mathscr{P} = p(r, z) + \rho g z$ and multiply both sides by -1.

$$0 = \frac{\partial \mathscr{P}}{\partial z}$$

This equation implies that even though the pressure may vary radially and vertically, the modified pressure as defined here only varies radially: $\mathscr{P} = \mathscr{P}(r)$. Write equation (8) in terms of this modified pressure now.

$$\begin{aligned} 0 &= -\frac{\partial}{\partial r}(p + \rho g z) + \mu \frac{\partial^2 v_r}{\partial z^2} \\ &= -\frac{\partial \mathscr{P}}{\partial r} + \mu \frac{\partial^2 v_r}{\partial z^2} \\ &= -\frac{d\mathscr{P}}{dr} + \mu \frac{\partial^2 v_r}{\partial z^2} \\ &= -\frac{d\mathscr{P}}{dr} + \mu \left(-\frac{2C_3}{b^2 r} \right) \\ &= -\frac{d\mathscr{P}}{dr} - \frac{2\mu C_3}{b^2 r} \end{aligned}$$

Solve this ODE for $\mathscr{P}$ by separating variables.

$$\frac{d\mathscr{P}}{dr} = -\frac{2\mu C_3}{b^2 r}$$

$$d\mathscr{P} = -\frac{2\mu C_3}{b^2 r}\, dr$$

Integrate both sides.

$$\int_{\mathscr{P}_1}^{\mathscr{P}_2} d\mathscr{P} = -\frac{2\mu C_3}{b^2} \int_{r_1}^{r_2} \frac{dr}{r}$$

$$\mathscr{P}_2 - \mathscr{P}_1 = -\frac{2\mu C_3}{b^2} \ln \frac{r_2}{r_1}$$

Solve for C_3.

$$C_3 = \frac{(\mathscr{P}_1 - \mathscr{P}_2) b^2}{2\mu \ln(r_2/r_1)}$$

Therefore,

$$v_r(r, z) = \frac{(\mathscr{P}_1 - \mathscr{P}_2) b^2}{2\mu r \ln(r_2/r_1)} \left[1 - \left(\frac{z}{b} \right)^2 \right].$$

Problem 3B.16

This problem considers a fluid flowing through two parallel plates with uniform cross flow.

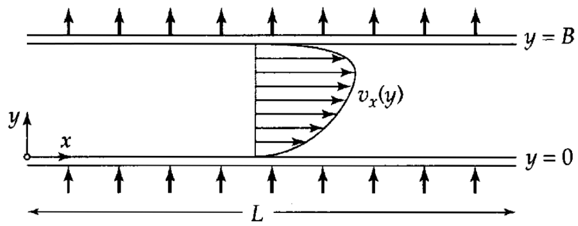

Fig. 3B.16. Flow in a slit of length L, width W, and thickness B. The walls at $y = 0$ and $y = B$ are porous, and there is a flow of the fluid in the y direction, with a uniform velocity $v_y = v_0$.

The velocity is assumed to have two components, one in the x-direction that varies with y and a second in the y-direction that is constant.

$$\mathbf{v} = v_x(y)\hat{\mathbf{x}} + v_0\hat{\mathbf{y}}$$

As a result, the vorticity only has a z-component.

$$\mathbf{w} = \nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_0 & 0 \end{vmatrix} = -\frac{dv_x}{dy}\hat{\mathbf{z}}$$

Substitute this into equation (1).

$$\underbrace{\frac{\partial}{\partial t}\left(-\frac{dv_x}{dy}\hat{\mathbf{z}}\right)}_{=\mathbf{0}} + \mathbf{v}\cdot\nabla\left(-\frac{dv_x}{dy}\hat{\mathbf{z}}\right) = \nu\nabla^2\left(-\frac{dv_x}{dy}\hat{\mathbf{z}}\right) + \left(-\frac{dv_x}{dy}\hat{\mathbf{z}}\right)\cdot\nabla\mathbf{v}$$

The components of $\mathbf{v}\cdot\nabla\mathbf{w}$ and $\mathbf{w}\cdot\nabla\mathbf{v}$ in Cartesian coordinates are given in the formulas at the bottom of page 832. Since v_x and v_y and w_z are nonzero, only the first two terms in equation (R) remain in the former and only the last term in equation (P) remains in the latter.

$$\left[\underbrace{v_x\frac{\partial}{\partial x}\left(-\frac{dv_x}{dy}\right)}_{=0} + v_0\frac{\partial}{\partial y}\left(-\frac{dv_x}{dy}\right)\right]\hat{\mathbf{z}} = \nu\nabla^2\left(-\frac{dv_x}{dy}\hat{\mathbf{z}}\right) + \underbrace{\left(-\frac{dv_x}{dy}\right)\frac{\partial v_x}{\partial z}\hat{\mathbf{x}}}_{=\mathbf{0}} + \underbrace{\left(-\frac{dv_x}{dy}\right)\frac{\partial v_0}{\partial z}\hat{\mathbf{y}}}_{=\mathbf{0}}$$

Because v_x is only a function of y and v_0 is constant, most of these terms are zero anyway. Lastly, use equation (O) on page 832 to write out the Laplacian.

$$-v_0\frac{d^2v_x}{dy^2}\hat{\mathbf{z}} = \nu\left[\underbrace{\frac{\partial^2}{\partial x^2}\left(-\frac{dv_x}{dy}\right)}_{=0} + \frac{\partial^2}{\partial y^2}\left(-\frac{dv_x}{dy}\right) + \underbrace{\frac{\partial^2}{\partial z^2}\left(-\frac{dv_x}{dy}\right)}_{=0}\right]\hat{\mathbf{z}}$$

Dot both sides by $\hat{\mathbf{z}}$ to get a scalar equation.

$$-v_0\frac{d^2v_x}{dy^2} = \nu\frac{d^2}{dy^2}\left(-\frac{dv_x}{dy}\right)$$

Divide both sides by $-\nu$.

$$\frac{v_0}{\nu}\frac{d^2v_x}{dy^2} = \frac{d^3v_x}{dy^3}$$

This is a linear homogeneous ODE with constant coefficients, so its general solution is of the form $v_x = e^{ry}$.

$$v_x = e^{ry} \quad \rightarrow \quad \frac{dv_x}{dy} = re^{ry} \quad \rightarrow \quad \frac{d^2v_x}{dy^2} = r^2e^{ry} \quad \rightarrow \quad \frac{d^3v_x}{dy^3} = r^3e^{ry}$$

Substitute these formulas into the ODE and solve for r.

$$\frac{v_0}{\nu}r^2e^{ry} = r^3e^{ry}$$

$$\frac{v_0}{\nu}r^2 = r^3$$

$$r^2\left(r - \frac{v_0}{\nu}\right) = 0$$

$$r = 0 \quad \text{or} \quad r = \frac{v_0}{\nu}$$

Two solutions to the ODE are $v_x = e^0 = 1$ and $v_x = e^{v_0y/\nu}$. Because the multiplicity of the $r = 0$ root is 2, a second linearly independent solution can be obtained from the first by including a factor of y: $v_x = ye^0 = y$. By the principle of superposition, the general solution is a linear combination of these three.

$$v_x(y) = C_1 + C_2y + C_3e^{v_0y/\nu}$$

Three boundary conditions are needed to determine C_1, C_2, and C_3. Two of them are obtained from the assumption that the fluid does not slip on the surfaces at $z = 0$ and $z = B$.

$$v_x(0) = 0$$
$$v_x(B) = 0$$

The third is from the fact that the fluid is flowing due to an imposed modified pressure gradient in the x-direction: At $x = 0$ the modified pressure is $\mathscr{P} = \mathscr{P}_0$, and at $x = L$ the modified pressure is $\mathscr{P} = \mathscr{P}_L$. Apply the first two boundary conditions.

$$v_x(0) = C_1 + C_3 = 0$$
$$v_x(B) = C_1 + C_2B + C_3e^{v_0B/\nu} = 0$$

Solving this system of equations for C_1 and C_3 yields

$$C_1 = \frac{BC_2}{e^{v_0B/\nu} - 1} \quad \text{and} \quad C_3 = -\frac{BC_2}{e^{v_0B/\nu} - 1},$$

which means the x-component of velocity becomes

$$\begin{aligned} v_x(y) &= \frac{BC_2}{e^{v_0B/\nu} - 1} + C_2y - \frac{BC_2}{e^{v_0B/\nu} - 1}e^{v_0y/\nu} \\ &= BC_2\left(\frac{1}{e^{v_0B/\nu} - 1} + \frac{y}{B} - \frac{e^{v_0y/\nu}}{e^{v_0B/\nu} - 1}\right) \\ &= BC_2\left(\frac{y}{B} - \frac{e^{v_0y/\nu} - 1}{e^{v_0B/\nu} - 1}\right). \end{aligned}$$

Since the modified pressure is involved, it's necessary to consider the Navier-Stokes equation.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla\cdot\rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}$$

From Appendix B.6 on page 848, it yields the following three scalar equations in Cartesian coordinates.

$$\begin{aligned}
\rho\left(\underbrace{\frac{\partial v_x}{\partial t}}_{=0} + \underbrace{v_x\frac{\partial v_x}{\partial x}}_{=0} + v_y\frac{\partial v_x}{\partial y} + \underbrace{v_z\frac{\partial v_x}{\partial z}}_{=0}\right) &= -\frac{\partial p}{\partial x} + \mu\left[\underbrace{\frac{\partial^2 v_x}{\partial x^2}}_{=0} + \frac{\partial^2 v_x}{\partial y^2} + \underbrace{\frac{\partial^2 v_x}{\partial z^2}}_{=0}\right] + \rho g_x \\
\rho\left(\underbrace{\frac{\partial v_y}{\partial t}}_{=0} + \underbrace{v_x\frac{\partial v_y}{\partial x}}_{=0} + \underbrace{v_y\frac{\partial v_y}{\partial y}}_{=0} + \underbrace{v_z\frac{\partial v_y}{\partial z}}_{=0}\right) &= -\frac{\partial p}{\partial y} + \mu\left[\underbrace{\frac{\partial^2 v_y}{\partial x^2}}_{=0} + \underbrace{\frac{\partial^2 v_y}{\partial y^2}}_{=0} + \underbrace{\frac{\partial^2 v_y}{\partial z^2}}_{=0}\right] + \rho g_y \\
\rho\left(\underbrace{\frac{\partial v_z}{\partial t}}_{=0} + \underbrace{v_x\frac{\partial v_z}{\partial x}}_{=0} + \underbrace{v_y\frac{\partial v_z}{\partial y}}_{=0} + \underbrace{v_z\frac{\partial v_z}{\partial z}}_{=0}\right) &= -\frac{\partial p}{\partial z} + \mu\left[\underbrace{\frac{\partial^2 v_z}{\partial x^2}}_{=0} + \underbrace{\frac{\partial^2 v_z}{\partial y^2}}_{=0} + \underbrace{\frac{\partial^2 v_z}{\partial z^2}}_{=0}\right] + \rho g_z
\end{aligned}$$

Gravity points in the x-direction, so $\mathbf{g} = g\hat{\mathbf{x}}$, which means $g_x = g$, $g_y = 0$, and $g_z = 0$.

$$\begin{aligned}
\rho v_0\frac{\partial v_x}{\partial y} &= -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 v_x}{\partial y^2} + \rho g \\
0 &= -\frac{\partial p}{\partial y} \\
0 &= -\frac{\partial p}{\partial z}
\end{aligned}$$

Combine the pressure and gravity terms in the first equation by introducing a modified pressure function.

$$\rho v_0\frac{\partial v_x}{\partial y} = -\frac{\partial}{\partial x}(p - \rho g x) + \mu\frac{\partial^2 v_x}{\partial y^2}$$

Let $\mathscr{P} = p(x) - \rho g x$

$$\rho v_0\frac{\partial v_x}{\partial y} = -\frac{\partial \mathscr{P}}{\partial x} + \mu\frac{\partial^2 v_x}{\partial y^2}$$

and write the y- and z-equations in terms of it.

$$\begin{aligned}
0 &= -\frac{\partial}{\partial y}(p - \rho g x) \quad \rightarrow \quad 0 = \frac{\partial \mathscr{P}}{\partial y} \\
0 &= -\frac{\partial}{\partial z}(p - \rho g x) \quad \rightarrow \quad 0 = \frac{\partial \mathscr{P}}{\partial z}
\end{aligned}$$

These equations imply that the modified pressure is only a function of x: $\mathscr{P} = \mathscr{P}(x)$. The previous equation then becomes

$$\rho v_0\frac{\partial v_x}{\partial y} = -\frac{d\mathscr{P}}{dx} + \mu\frac{\partial^2 v_x}{\partial y^2}$$

$$\rho v_0 B C_2\left(\frac{1}{B} - \frac{v_0}{\nu}\frac{e^{v_0 y/\nu}}{e^{v_0 B/\nu} - 1}\right) = -\frac{\mathscr{P}_L - \mathscr{P}_0}{L - 0} + \mu B C_2\left(-\frac{v_0^2}{\nu^2}\frac{e^{v_0 y/\nu}}{e^{v_0 B/\nu} - 1}\right).$$

Solve for C_2, noting that the kinematic viscosity is $\nu = \mu/\rho$.

$$BC_2\left(\frac{\rho v_0}{B} - \frac{\rho v_0^2}{\nu}\frac{e^{v_0 y/\nu}}{e^{v_0 B/\nu}-1} + \mu\frac{v_0^2}{\nu^2}\frac{e^{v_0 y/\nu}}{e^{v_0 B/\nu}-1}\right) = \frac{\mathscr{P}_0 - \mathscr{P}_L}{L}$$

$$C_2\left(\rho v_0 - \frac{\rho v_0^2 B}{\nu}\frac{e^{v_0 y/\nu}}{e^{v_0 B/\nu}-1} + \mu\frac{v_0^2 B}{\nu^2}\frac{e^{v_0 y/\nu}}{e^{v_0 B/\nu}-1}\right) = \frac{\mathscr{P}_0 - \mathscr{P}_L}{L}$$

$$C_2\left(\rho v_0 - \frac{\rho^2 v_0^2 B}{\mu}\frac{e^{v_0 y/\nu}}{e^{v_0 B/\nu}-1} + \frac{\rho^2 v_0^2 B}{\mu}\frac{e^{v_0 y/\nu}}{e^{v_0 B/\nu}-1}\right) = \frac{\mathscr{P}_0 - \mathscr{P}_L}{L}$$

$$C_2 \rho v_0 = \frac{\mathscr{P}_0 - \mathscr{P}_L}{L}$$

$$C_2 = \frac{\mathscr{P}_0 - \mathscr{P}_L}{\rho v_0 L}$$

Therefore,

$$\begin{aligned} v_x(y) &= BC_2\left(\frac{y}{B} - \frac{e^{v_0 y/\nu}-1}{e^{v_0 B/\nu}-1}\right) \\ &= \frac{(\mathscr{P}_0 - \mathscr{P}_L)B}{\rho v_0 L}\left(\frac{y}{B} - \frac{e^{v_0 y/\nu}-1}{e^{v_0 B/\nu}-1}\right) \\ &= \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2}{\mu L}\frac{\mu}{\rho v_0 B}\left(\frac{y}{B} - \frac{e^{\rho v_0 y/\mu}-1}{e^{\rho v_0 B/\mu}-1}\right) \\ &= \frac{(\mathscr{P}_0 - \mathscr{P}_L)B^2}{\mu L}\frac{1}{A}\left(\frac{y}{B} - \frac{e^{Ay/B}-1}{e^{A}-1}\right), \end{aligned}$$

where $A = \rho v_0 B/\mu$.