Problem 4B.6

Potential flow near a stagnation point (Fig. 4B.6).

(a) Show that the complex potential $w = -v_0 z^2$ describes the flow near a plane stagnation point.

(b) Find the velocity components $v_x(x, y)$ and $v_y(x, y)$.

(c) Explain the physical significance of v_0.

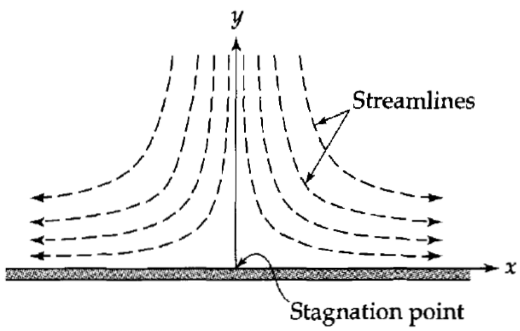


Fig. 4B.6. Two-dimensional potential flow near a stagnation point.

Solution

Part (a)

Substitute $z = x + iy$, and write w as the sum of a real part and an imaginary part (in other words, put w in rectangular form).

$$\begin{aligned} w &= -v_0 z^2 \\ &= -v_0(x + iy)^2 \\ &= -v_0(x^2 + 2ixy + i^2y^2) \\ &= -v_0(x^2 - y^2 + 2ixy) \\ &= v_0(y^2 - x^2) - 2iv_0xy \\ &= \phi(x, y) + i\psi(x, y) \end{aligned}$$

The real part of w is the velocity potential ϕ, and the imaginary part of w is the stream function.

$$\begin{aligned} \phi(x, y) &= v_0(y^2 - x^2) \\ \psi(x, y) &= -2v_0xy \end{aligned}$$

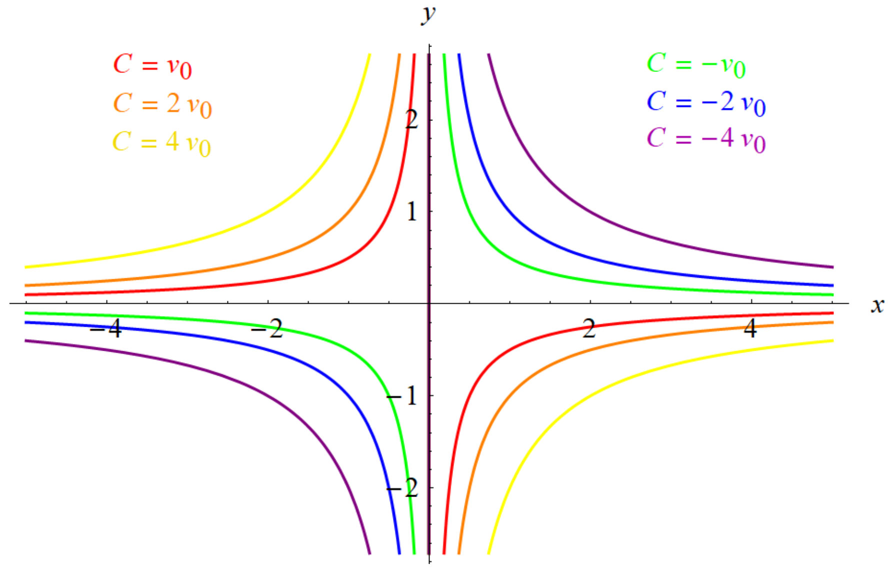
In order to see what kind of flow w models, plot the streamlines $\psi(x, y) = C$ for various values of the constant C.

$$\psi(x, y) = C$$

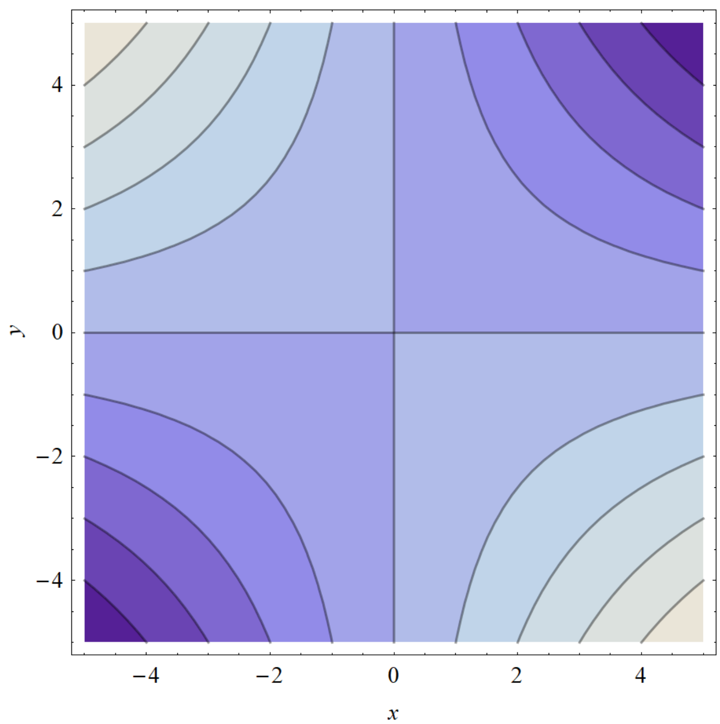
$$-2v_0xy = C$$

$$y(x) = -\frac{C}{2v_0x}$$

As illustrated in the graph below, positive values of C yield streamlines in the second and fourth quadrants, and negative values of C yield streamlines in the first and third quadrants.



Instead of plotting the function manually for several values of C, it's more convenient to make a contour plot of the stream function with $v_0 = 1$.



$w = -v_0 z^2$ does indeed describe the flow near a horizontal or vertical plane with a stagnation point at the origin.

Part (b)

The velocity is obtained by taking the negative gradient of the velocity potential.

$$\begin{aligned}\mathbf{v} &= -\nabla\phi \\ &= -\left[\frac{\partial}{\partial x}v_0(y^2 - x^2)\hat{\mathbf{x}} + \frac{\partial}{\partial y}v_0(y^2 - x^2)\hat{\mathbf{y}}\right] \\ &= -\left[v_0(-2x)\hat{\mathbf{x}} + v_0(2y)\hat{\mathbf{y}}\right] \\ &= 2v_0 x\,\hat{\mathbf{x}} - 2v_0 y\,\hat{\mathbf{y}}\end{aligned}$$

Therefore,

$$\begin{aligned}v_x(x) &= 2v_0 x \\ v_y(y) &= -2v_0 y\end{aligned}.$$

Part (c)

Determine the magnitude of $\mathbf{v}$ (also known as the speed).

$$|\mathbf{v}| = \sqrt{(2v_0 x)^2 + (-2v_0 y)^2} = \sqrt{4v_0^2 x^2 + 4v_0^2 y^2} = 2v_0\sqrt{x^2 + y^2}$$

v_0 is a parameter that measures how fast the fluid is flowing.