Problem 4B.9

Checking solutions to unsteady flow problems.

(a) Verify the solutions to the problems in Examples 4.1-1, 2, and 3 by showing that they satisfy the partial differential equations, initial conditions, and boundary conditions. To show that Eq. 4.1-15 satisfies the differential equation, one has to know how to differentiate an integral using the *Leibniz formula* given in §C.3.

(b) In Example 4.1-3 the initial condition is not satisfied by Eq. 4.1-57. Why?

Solution

Example 4.1-1: Flow near a Wall Suddenly Set in Motion

The initial boundary value problem solved in Example 4.1-1 read as follows.

PDE:		$\frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial y^2}$		(4.1-1)
I.C.:	at $t \le 0$,	$v_x = 0$	for all y	(4.1-2)
B.C. 1:	at $y = 0$,	$v_x = v_0$	for all $t > 0$	(4.1-3)
B.C. 2:	at $y = \infty$,	$v_x = 0$	for all $t > 0$	(4.1-4)

The solution was found to be in terms of the complementary error function.

$$\begin{aligned} v_x(y,t) &= v_0 \operatorname{erfc} \frac{y}{\sqrt{4\nu t}} \\ &= v_0 \left(1 - \operatorname{erf} \frac{y}{\sqrt{4\nu t}}\right) \\ &= v_0 \left(1 - \frac{2}{\sqrt{\pi}} \int_0^{y/\sqrt{4\nu t}} e^{-\eta^2}\, d\eta\right) \end{aligned}$$

If we set $y = 0$ in the formula for v_x,

$$\begin{aligned} v_x(0,t) &= v_0 \left(1 - \frac{2}{\sqrt{\pi}} \int_0^0 e^{-\eta^2}\, d\eta\right) \\ &= v_0(1-0) \\ &= v_0 \end{aligned}$$

then B.C. 1 is satisfied. Setting either $t = 0$ or $y = \infty$ in the formula for v_x results in

$$\begin{aligned} v_x(y,0) = v_x(\infty,t) &= v_0 \left(1 - \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-\eta^2}\, d\eta\right) \\ &= v_0 \left(1 - \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2}\right) \\ &= v_0(1-1) \\ &= 0, \end{aligned}$$

so I.C. and B.C. 2 are satisfied. All that's left to do is to check that the PDE is satisfied. Find the first derivative of v_x with respect to t.

$$\begin{aligned}
\frac{\partial v_x}{\partial t} &= v_0 \frac{\partial}{\partial t}\left(1 - \frac{2}{\sqrt{\pi}}\int_0^{y/\sqrt{4\nu t}} e^{-\eta^2}\,d\eta\right) \\
&= v_0\left(-\frac{2}{\sqrt{\pi}}\right)\frac{\partial}{\partial t}\int_0^{y/\sqrt{4\nu t}} e^{-\eta^2}\,d\eta \\
&= v_0\left(-\frac{2}{\sqrt{\pi}}\right)\exp\left(-\frac{y^2}{4\nu t}\right)\frac{\partial}{\partial t}\left(\frac{y}{\sqrt{4\nu t}}\right) \\
&= v_0\left(-\frac{2}{\sqrt{\pi}}\right)\exp\left(-\frac{y^2}{4\nu t}\right)\left(-\frac{1}{2}\frac{y}{\sqrt{4\nu t^3}}\right) \\
&= \frac{v_0 y}{\sqrt{4\pi\nu t^3}}\exp\left(-\frac{y^2}{4\nu t}\right)
\end{aligned}$$

Now find the second derivative of v_x with respect to y.

$$\begin{aligned}
\frac{\partial v_x}{\partial y} &= v_0 \frac{\partial}{\partial y}\left(1 - \frac{2}{\sqrt{\pi}}\int_0^{y/\sqrt{4\nu t}} e^{-\eta^2}\,d\eta\right) \\
&= v_0\left(-\frac{2}{\sqrt{\pi}}\right)\frac{\partial}{\partial y}\int_0^{y/\sqrt{4\nu t}} e^{-\eta^2}\,d\eta \\
&= v_0\left(-\frac{2}{\sqrt{\pi}}\right)\exp\left(-\frac{y^2}{4\nu t}\right)\frac{\partial}{\partial y}\left(\frac{y}{\sqrt{4\nu t}}\right) \\
&= v_0\left(-\frac{2}{\sqrt{\pi}}\right)\exp\left(-\frac{y^2}{4\nu t}\right)\left(\frac{1}{\sqrt{4\nu t}}\right) \\
&= -\frac{v_0}{\sqrt{\pi\nu t}}\exp\left(-\frac{y^2}{4\nu t}\right) \\
\frac{\partial^2 v_x}{\partial y^2} &= -\frac{v_0}{\sqrt{\pi\nu t}}\frac{\partial}{\partial y}\exp\left(-\frac{y^2}{4\nu t}\right) \\
&= -\frac{v_0}{\sqrt{\pi\nu t}}\exp\left(-\frac{y^2}{4\nu t}\right)\frac{\partial}{\partial y}\left(-\frac{y^2}{4\nu t}\right) \\
&= -\frac{v_0}{\sqrt{\pi\nu t}}\exp\left(-\frac{y^2}{4\nu t}\right)\left(-\frac{y}{2\nu t}\right) \\
&= \frac{v_0 y}{\nu\sqrt{4\pi\nu t^3}}\exp\left(-\frac{y^2}{4\nu t}\right)
\end{aligned}$$

Therefore, the solution satisfies the PDE,

$$\frac{\partial v_x}{\partial t} = \nu\frac{\partial^2 v_x}{\partial y^2}.$$

Example 4.1-2: Unsteady Laminar Flow Between Two Parallel Plates

The initial boundary value problem solved in Example 4.1-2 read as follows.

PDE:		$\frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial y^2}$		(4.1-16)
I.C.:	at $t \leq 0$,	$v_x = 0$	for $0 \leq y \leq b$	(4.1-17)
B.C. 1:	at $y = 0$,	$v_x = v_0$	for all $t > 0$	(4.1-18)
B.C. 2:	at $y = b$,	$v_x = 0$	for all $t > 0$	(4.1-19)

The solution was found to be the sum of a steady part and a transient part.

$$v_x(y,t) = v_0 \left[1 - \frac{y}{b} - \sum_{n=1}^{\infty} \left(\frac{2}{n\pi}\right) \exp\left(-n^2\pi^2 \frac{\nu t}{b^2}\right) \sin \frac{n\pi y}{b}\right]$$

If we set $y = 0$ in the formula for v_x,

$$\begin{aligned} v_x(0,t) &= v_0 \left[1 - \frac{0}{b} - \sum_{n=1}^{\infty} \left(\frac{2}{n\pi}\right) \exp\left(-n^2\pi^2 \frac{\nu t}{b^2}\right) \sin 0\right] \\ &= v_0(1-0) \\ &= v_0 \end{aligned}$$

then B.C. 1 is satisfied. If we set $y = b$ in the formula for v_x,

$$\begin{aligned} v_x(0,t) &= v_0 \left[1 - \frac{b}{b} - \sum_{n=1}^{\infty} \left(\frac{2}{n\pi}\right) \exp\left(-n^2\pi^2 \frac{\nu t}{b^2}\right) \underbrace{\sin n\pi}_{=0}\right] \\ &= v_0(1-1) \\ &= 0 \end{aligned}$$

then B.C. 2 is satisfied. If we set $t = 0$ in the formula for v_x, then

$$v_x(y,0) = v_0 \left[1 - \frac{y}{b} - \sum_{n=1}^{\infty} \left(\frac{2}{n\pi}\right) \sin \frac{n\pi y}{b}\right].$$

For the right side to be zero, it must be verified that

$$\sum_{n=1}^{\infty} \left(\frac{2}{n\pi}\right) \sin \frac{n\pi y}{b} \stackrel{?}{=} 1 - \frac{y}{b}.$$

Multiply both sides by $\sin m\pi y/b$, where m is another integer.

$$\sum_{n=1}^{\infty} \left(\frac{2}{n\pi}\right) \sin \frac{n\pi y}{b} \sin \frac{m\pi y}{b} \stackrel{?}{=} \left(1 - \frac{y}{b}\right) \sin \frac{m\pi y}{b}$$

Integrate both sides with respect to y from 0 to b.

$$\int_0^b \sum_{n=1}^{\infty} \left(\frac{2}{n\pi}\right) \sin \frac{n\pi y}{b} \sin \frac{m\pi y}{b}\, dy \stackrel{?}{=} \int_0^b \left(1 - \frac{y}{b}\right) \sin \frac{m\pi y}{b}\, dy$$

Bring the constants in front of the integrals.

$$\sum_{n=1}^{\infty}\left(\frac{2}{n\pi}\right)\int_0^b \sin\frac{n\pi y}{b}\sin\frac{m\pi y}{b}\,dy \stackrel{?}{=} \int_0^b\left(1-\frac{y}{b}\right)\sin\frac{m\pi y}{b}\,dy$$

If $n \neq m$, then the integral on the left side is zero (as can be verified with the product-to-sum formula for sine). Only if $n = m$ does the integral yield a nonzero result, which means only one term remains in the infinite series.

$$\left(\frac{2}{n\pi}\right)\int_0^b \sin^2\frac{n\pi y}{b}\,dy \stackrel{?}{=} \int_0^b\left(1-\frac{y}{b}\right)\sin\frac{n\pi y}{b}\,dy$$

Evaluate the integrals.

$$\begin{aligned}\left(\frac{2}{n\pi}\right)\frac{b}{2} &\stackrel{?}{=} \frac{b(n\pi - \overbrace{\sin n\pi}^{=0})}{n^2\pi^2}\\ \frac{b}{n\pi} &= \frac{b}{n\pi}\end{aligned}$$

Therefore, $v_x(y,0) = 0$ and I.C. is satisfied. All that's left to do is to check that v_x satisfies the PDE. Find the first derivative of v_x with respect to t.

$$\begin{aligned}\frac{\partial v_x}{\partial t} &= v_0\frac{\partial}{\partial t}\left[1-\frac{y}{b}-\sum_{n=1}^{\infty}\left(\frac{2}{n\pi}\right)\exp\left(-n^2\pi^2\frac{\nu t}{b^2}\right)\sin\frac{n\pi y}{b}\right]\\ &= v_0\left[-\sum_{n=1}^{\infty}\left(\frac{2}{n\pi}\right)\frac{\partial}{\partial t}\exp\left(-n^2\pi^2\frac{\nu t}{b^2}\right)\sin\frac{n\pi y}{b}\right]\\ &= v_0\left[-\sum_{n=1}^{\infty}\left(\frac{2}{n\pi}\right)\exp\left(-n^2\pi^2\frac{\nu t}{b^2}\right)\left(-n^2\pi^2\frac{\nu}{b^2}\right)\sin\frac{n\pi y}{b}\right]\\ &= \nu\sum_{n=1}^{\infty}\left(\frac{2v_0 n\pi}{b^2}\right)\exp\left(-n^2\pi^2\frac{\nu t}{b^2}\right)\sin\frac{n\pi y}{b}\end{aligned}$$

Now find the second derivative of v_x with respect to y.

$$\begin{aligned}\frac{\partial^2 v_x}{\partial y^2} &= v_0\frac{\partial^2}{\partial y^2}\left[1-\frac{y}{b}-\sum_{n=1}^{\infty}\left(\frac{2}{n\pi}\right)\exp\left(-n^2\pi^2\frac{\nu t}{b^2}\right)\sin\frac{n\pi y}{b}\right]\\ &= v_0\left[-\sum_{n=1}^{\infty}\left(\frac{2}{n\pi}\right)\exp\left(-n^2\pi^2\frac{\nu t}{b^2}\right)\frac{\partial^2}{\partial y^2}\sin\frac{n\pi y}{b}\right]\\ &= v_0\left[-\sum_{n=1}^{\infty}\left(\frac{2}{n\pi}\right)\exp\left(-n^2\pi^2\frac{\nu t}{b^2}\right)\left(-\frac{n^2\pi^2}{b^2}\right)\sin\frac{n\pi y}{b}\right]\\ &= \sum_{n=1}^{\infty}\left(\frac{2v_0 n\pi}{b^2}\right)\exp\left(-n^2\pi^2\frac{\nu t}{b^2}\right)\sin\frac{n\pi y}{b}\end{aligned}$$

Therefore, the solution satisfies the PDE,

$$\frac{\partial v_x}{\partial t} = \nu\frac{\partial^2 v_x}{\partial y^2}.$$

Example 4.1-3: Unsteady Laminar Flow near an Oscillating Plate

The initial boundary value problem solved in Example 4.1-2 read as follows.

$$\text{PDE:} \qquad \frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial y^2} \tag{4.1-44}$$

$$\text{I.C.:} \qquad \text{at } t \le 0, \qquad v_x = 0 \qquad \text{for all } y \tag{4.1-45}$$

$$\text{B.C. 1:} \qquad \text{at } y = 0, \qquad v_x = v_0 \Re\{e^{i\omega t}\} \qquad \text{for all } t > 0 \tag{4.1-46}$$

$$\text{B.C. 2:} \qquad \text{at } y = \infty, \qquad v_x = 0 \qquad \text{for all } t > 0 \tag{4.1-47}$$

The asymptotic solution (valid for large times) was found to be

$$v_x(y,t) = v_0 e^{-\sqrt{\omega/2\nu}y} \cos(\omega t - \sqrt{\omega/2\nu}y). \tag{4.1-57}$$

If we set $y = 0$ in the formula for v_x,

$$\begin{aligned} v_x(0,t) &= v_0 e^0 \cos(\omega t - 0) \\ &= v_0 \cos \omega t \\ &= v_0 \Re\{e^{i\omega t}\} \end{aligned}$$

then B.C. 1 is satisfied. If we set $y = \infty$ in the formula for v_x,

$$\begin{aligned} v_x(\infty,t) &= v_0 e^{-\infty} \cos(-\infty) \\ &= 0 \end{aligned}$$

then B.C. 2 is satisfied. The initial condition is not satisfied because the formula for v_x is only valid as $t \to \infty$. (See Problem 4D.1 for the complete solution.) All that's left to do is to check that v_x satisfies the PDE. Find the first derivative of v_x with respect to t.

$$\frac{\partial v_x}{\partial t} = -\omega v_0 e^{-\sqrt{\omega/2\nu}y} \sin(\omega t - \sqrt{\omega/2\nu}y)$$

Now find the second derivative of v_x with respect to y.

$$\begin{aligned} \frac{\partial v_x}{\partial y} &= -v_0 \sqrt{\frac{\omega}{2\nu}} e^{-\sqrt{\omega/2\nu}y} \cos(\omega t - \sqrt{\omega/2\nu}y) + v_0 \sqrt{\frac{\omega}{2\nu}} e^{-\sqrt{\omega/2\nu}y} \sin(\omega t - \sqrt{\omega/2\nu}y) \\ &= v_0 \sqrt{\frac{\omega}{2\nu}} e^{-\sqrt{\omega/2\nu}y} [\sin(\omega t - \sqrt{\omega/2\nu}y) - \cos(\omega t - \sqrt{\omega/2\nu}y)] \\ \frac{\partial^2 v_x}{\partial y^2} &= -v_0 \frac{\omega}{2\nu} e^{-\sqrt{\omega/2\nu}y} [\sin(\omega t - \sqrt{\omega/2\nu}y) - \cancel{\cos(\omega t - \sqrt{\omega/2\nu}y)}] \\ &\qquad + v_0 \sqrt{\frac{\omega}{2\nu}} e^{-\sqrt{\omega/2\nu}y} [\cancel{\cos(\omega t - \sqrt{\omega/2\nu}y)} + \sin(\omega t - \sqrt{\omega/2\nu}y)] \left(-\sqrt{\frac{\omega}{2\nu}}\right) \\ &= -v_0 \frac{\omega}{2\nu} e^{-\sqrt{\omega/2\nu}y} [2 \sin(\omega t - \sqrt{\omega/2\nu}y)] \\ &= -\frac{\omega v_0}{\nu} e^{-\sqrt{\omega/2\nu}y} \sin(\omega t - \sqrt{\omega/2\nu}y) \end{aligned}$$

Therefore, the solution satisfies the PDE,

$$\frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial y^2}.$$