Problem 4D.2

Start-up of laminar flow in a circular tube (Fig. 4D.2). A fluid of constant density and viscosity is contained in a very long pipe of length L and radius R. Initially the fluid is at rest. At time $t = 0$, a pressure gradient $(\mathscr{P}_0 - \mathscr{P}_L)/L$ is imposed on the system. Determine how the velocity profiles change with time.

(a) Show that the relevant equation of motion can be put into dimensionless form as follows:

$$\frac{\partial \phi}{\partial \tau} = 4 + \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi \frac{\partial \phi}{\partial \xi}\right) \tag{4D.2-1}$$

in which $\xi = r/R$, $\tau = \mu t/\rho R^2$, and $\phi = [(\mathscr{P}_0 - \mathscr{P}_L)R^2/4\mu L]^{-1} v_z$.

(b) Show that the asymptotic solution for large time is $\phi_\infty = 1 - \xi^2$. Then define ϕ_t by $\phi(\xi, \tau) = \phi_\infty(\xi) - \phi_t(\xi, \tau)$, and solve the partial differential equation for ϕ_t by the method of separation of variables.

(c) Show that the final solution is

$$\phi(\xi, \tau) = (1 - \xi^2) - 8 \sum_{n=1}^{\infty} \frac{J_0(\alpha_n \xi)}{\alpha_n^3 J_1(\alpha_n)} \exp(-\alpha_n^2 \tau) \tag{4D.2-2}$$

in which $J_n(\xi)$ is the nth order Bessel function of ξ, and the α_n are the roots of the equation $J_0(\alpha_n) = 0$. The result is plotted in Fig. 4D.2.

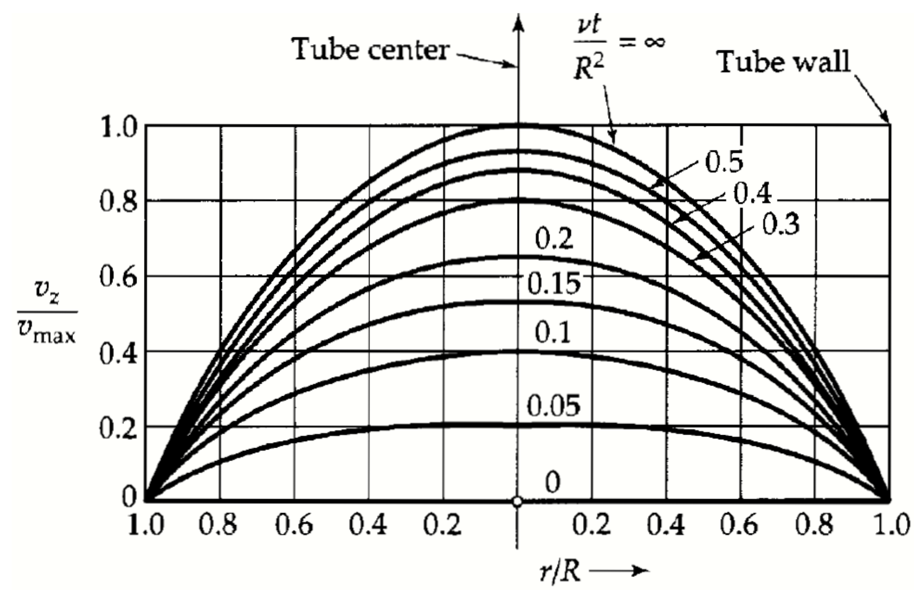

Fig. 4D.2. Velocity distribution for the unsteady flow resulting from a suddenly impressed pressure gradient in a circular tube [P. Szymanski, *J. Math. Pures Appl.*, Series 9, **11**, 67–107 (1932)].

Solution

Part (a)

Since the flow is unsteady and occuring along the axis of a circular tube, we assume that the velocity varies as a function of radius and time and that the fluid moves only in the z-direction.

$$\mathbf{v} = v_z(r,t)\hat{\mathbf{z}}$$

If we assume the fluid does not slip on the tube wall, then it has the wall's velocity at $r = R$.

$$\text{Boundary Condition 1:} \qquad v_z(R,t) = 0$$

As a consequence, the velocity at the tube center (the point furthest away from the wall) is a maximum.

$$\text{Boundary Condition 2:} \qquad \frac{\partial v_z}{\partial r}(0,t) = 0$$

The fluid starts from rest, so the initial velocity is zero.

$$\text{Initial Condition:} \qquad v_z(r,0) = 0$$

The equation of continuity results by considering a mass balance over a volume element that the fluid is flowing through. The fluid density ρ is constant, so the equation simplifies to

$$\nabla \cdot \mathbf{v} = 0.$$

The equation of motion results by considering a momentum balance over a volume element that the fluid is flowing through. ρ and the fluid viscosity μ are constant, so it simplifies to the Navier-Stokes equation.

$$\frac{\partial}{\partial t}\rho\mathbf{v} + \nabla \cdot \rho\mathbf{v}\mathbf{v} = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}$$

As this is a vector equation, it actually represents three scalar equations—one for each variable in the chosen coordinate system. Using cylindrical coordinates is the appropriate choice for this problem, so the two previous equations will be used in (r,θ,z). From Appendix B.4 on page 846, the continuity equation becomes

$$\underbrace{\frac{1}{r}\frac{\partial}{\partial r}(rv_r)}_{=0} + \underbrace{\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}}_{=0} + \underbrace{\frac{\partial v_z}{\partial z}}_{=0} = 0,$$

which doesn't tell us anything. From Appendix B.6 on page 848, the Navier-Stokes equation yields the following three scalar equations in cylindrical coordinates.

$$\rho\left(\underbrace{\frac{\partial v_r}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_r}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_r}{\partial z}}_{=0} - \underbrace{\frac{v_\theta^2}{r}}_{=0}\right) = -\underbrace{\frac{\partial p}{\partial r}}_{=0} + \mu\left[\underbrace{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)}_{=0} + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_r}{\partial z^2}}_{=0} - \underbrace{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}}_{=0}\right] + \underbrace{\rho g_r}_{=0}$$

$$\rho\left(\underbrace{\frac{\partial v_\theta}{\partial t}}_{=0} + \underbrace{v_r\frac{\partial v_\theta}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_\theta}{\partial z}}_{=0} + \underbrace{\frac{v_r v_\theta}{r}}_{=0}\right) = -\underbrace{\frac{1}{r}\frac{\partial p}{\partial \theta}}_{=0} + \mu\left[\underbrace{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right)}_{=0} + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_\theta}{\partial z^2}}_{=0} + \underbrace{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}}_{=0}\right] + \underbrace{\rho g_\theta}_{=0}$$

$$\rho\left(\frac{\partial v_z}{\partial t} + \underbrace{v_r\frac{\partial v_z}{\partial r}}_{=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}}_{=0} + \underbrace{v_z\frac{\partial v_z}{\partial z}}_{=0}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 v_z}{\partial z^2}}_{=0}\right] + \rho g_z$$

The relevant equation for the velocity is the z-equation, which has simplified considerably from the assumption that $\mathbf{v} = v_z\hat{\mathbf{z}}$.

$$\rho\frac{\partial v_z}{\partial t} = -\frac{\partial p}{\partial z} + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \rho g_z$$

$-\partial p/\partial z$ is the pressure gradient $-(p_L - p_0)/(L-0)$, and $-\partial p/\partial z + \rho g_z$ is the modified pressure gradient $-(\mathscr{P}_L - \mathscr{P}_0)/(L-0)$, which is given.

$$\rho\frac{\partial v_z}{\partial t} = \frac{\mathscr{P}_0 - \mathscr{P}_L}{L} + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)$$

The aim now is to put the partial differential equation into dimensionless form. Multiply both sides by $4L/(\mathscr{P}_0 - \mathscr{P}_L)$.

$$\rho\frac{\partial v_z}{\partial t}\frac{4L}{\mathscr{P}_0 - \mathscr{P}_L} = 4 + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\frac{4\mu L}{\mathscr{P}_0 - \mathscr{P}_L}\right)$$

Introduce R^2 in the numerator and denominator on both sides.

$$\rho R^2\frac{\partial v_z}{\partial t}\frac{4L}{(\mathscr{P}_0 - \mathscr{P}_L)R^2} = 4 + \frac{R}{r}R\frac{\partial}{\partial r}\left[r\frac{\partial v_z}{\partial r}\frac{4\mu L}{(\mathscr{P}_0 - \mathscr{P}_L)R^2}\right]$$

Introduce μ in the numerator and denominator on the left side, and introduce R in the numerator and denominator on the right side.

$$\frac{\rho R^2}{\mu}\frac{\partial v_z}{\partial t}\frac{4\mu L}{(\mathscr{P}_0 - \mathscr{P}_L)R^2} = 4 + \frac{R}{r}R\frac{\partial}{\partial r}\left[\frac{r}{R}R\frac{\partial v_z}{\partial r}\frac{4\mu L}{(\mathscr{P}_0 - \mathscr{P}_L)R^2}\right]$$

Here the dependent variable will be changed. Let

$$\phi = v_z\frac{4\mu L}{(\mathscr{P}_0 - \mathscr{P}_L)R^2}.$$

Notice that the fraction is the reciprocal of the maximum velocity for flow in a circular tube, so ϕ can be written as v_z/v_{max}, a dimensionless velocity. The independent variables will also be changed to ones that are dimensionless.

$$\xi = \frac{r}{R} \quad\rightarrow\quad d\xi = \frac{dr}{R} \quad\Rightarrow\quad R\frac{\partial}{\partial r} = \frac{\partial}{\partial \xi}$$

$$\tau = \frac{\mu t}{\rho R^2} \quad\rightarrow\quad d\tau = \frac{\mu\, dt}{\rho R^2} \quad\Rightarrow\quad \frac{\rho R^2}{\mu}\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau}$$

Therefore, the governing differential equation for velocity with dimensionless variables is

$$\frac{\partial \phi}{\partial \tau} = 4 + \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi\frac{\partial \phi}{\partial \xi}\right). \tag{1}$$

In terms of the new variables, the boundary and initial conditions become

$$\phi(1,\tau) = 0 \tag{2}$$

$$\frac{\partial \phi}{\partial \xi}(0,\tau) = 0 \tag{3}$$

$$\phi(\xi,0) = 0. \tag{4}$$

Part (b)

As a result of the imposed pressure gradient, the fluid starts to move and eventually reaches a steady state. The velocity can be thought to have an equilibrium part $\phi_\infty(\xi)$, independent of time, and a transient part $\phi_t(\xi,\tau)$: $\phi(\xi,\tau) = \phi_\infty(\xi) - \phi_t(\xi,\tau)$. At equilibrium ($t \to \infty$) the velocity satisfies

$$0 = 4 + \frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{d\phi_\infty}{d\xi}\right) \quad \to \quad \frac{d}{d\xi}\left(\xi\frac{d\phi_\infty}{d\xi}\right) = -4\xi.$$

Integrate both sides with respect to ξ.

$$\xi\frac{d\phi_\infty}{d\xi} = -2\xi^2 + C_1$$

Divide both sides by ξ.

$$\frac{d\phi_\infty}{d\xi} = -2\xi + \frac{C_1}{\xi}$$

In order for equation (3) to be satisfied, we require that $C_1 = 0$. Integrate both sides with respect to ξ once more.

$$\phi_\infty(\xi) = -\xi^2 + C_2$$

Apply equation (2) here to determine C_2.

$$\phi_\infty(1) = -1 + C_2 = 0 \quad \to \quad C_2 = 1$$

Therefore, the steady-state solution is

$$\boxed{\phi_\infty(\xi) = 1 - \xi^2.}$$

Now make the substitution $\phi(\xi,\tau) = 1 - \xi^2 - \phi_t(\xi,\tau)$ in equation (1).

$$\begin{aligned}
\frac{\partial}{\partial\tau}[1-\xi^2-\phi_t(\xi,\tau)] &= 4 + \frac{1}{\xi}\frac{\partial}{\partial\xi}\left\{\xi\frac{\partial}{\partial\xi}\left[1-\xi^2-\phi_t(\xi,\tau)\right]\right\} \\
-\frac{\partial\phi_t}{\partial\tau} &= 4 + \frac{1}{\xi}\frac{\partial}{\partial\xi}\left[\xi\left(-2\xi - \frac{\partial\phi_t}{\partial\xi}\right)\right] \\
-\frac{\partial\phi_t}{\partial\tau} &= 4 + \frac{1}{\xi}\frac{\partial}{\partial\xi}\left(-2\xi^2 - \xi\frac{\partial\phi_t}{\partial\xi}\right) \\
-\frac{\partial\phi_t}{\partial\tau} &= 4 + \frac{1}{\xi}\left[-4\xi - \frac{\partial}{\partial\xi}\left(\xi\frac{\partial\phi_t}{\partial\xi}\right)\right] \\
-\frac{\partial\phi_t}{\partial\tau} &= \not{4} - \not{4} - \frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\phi_t}{\partial\xi}\right)
\end{aligned}$$

Multiplying both sides by -1, we obtain a homogeneous partial differential equation for ϕ_t.

$$\frac{\partial\phi_t}{\partial\tau} = \frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\phi_t}{\partial\xi}\right)$$

Because the boundary conditions associated with it, $\frac{\partial\phi_t}{\partial\xi}(0,\tau) = 0$ and $\phi_t(1,\tau) = 0$, are homogeneous as well, the method of separation of variables can be applied to solve the equation. Assume a product solution of the form $\phi_t(\xi,\tau) = X(\xi)T(\tau)$ and plug it into the PDE

$$\frac{\partial}{\partial\tau}[X(\xi)T(\tau)] = \frac{1}{\xi}\frac{\partial}{\partial\xi}\left\{\xi\frac{\partial}{\partial\xi}[X(\xi)T(\tau)]\right\} \quad \to \quad XT' = \frac{T}{\xi}\frac{d}{d\xi}\left(\xi X'\right)$$

and the boundary conditions.

$$\begin{aligned} \frac{\partial \phi_t}{\partial \xi}(0,\tau) = 0 \quad &\rightarrow \quad X'(0)T(t) = 0 \quad &\rightarrow \quad X'(0) = 0 \\ \phi_t(1,\tau) = 0 \quad &\rightarrow \quad X(1)T(t) = 0 \quad &\rightarrow \quad X(1) = 0 \end{aligned}$$

Now separate variables in the PDE: bring all functions of τ to the left side and all functions of ξ to the right side.

$$\frac{T'}{T} = \frac{1}{\xi X}\frac{d}{d\xi}\left(\xi X'\right)$$

The only way a function of τ on the left can be equal to a function of ξ on the right is if both sides are equal to a constant λ.

$$\frac{T'}{T} = \frac{1}{\xi X}\frac{d}{d\xi}\left(\xi X'\right) = \lambda$$

Values of λ for which $X'(0) = 0$ and $X(1) = 0$ are satisfied are called the eigenvalues, and the nontrivial functions $X(\xi)$ associated with them are called the eigenfunctions.

Determination of Positive Eigenvalues: $\lambda = \eta^2$

Assuming λ is positive, the differential equation for T becomes

$$\frac{T'}{T} = \eta^2.$$

Multiply both sides by T.

$$T' = \eta^2 T$$

The general solution can be written in terms of the exponential function.

$$T(\tau) = C_3 e^{\eta^2 \tau}$$

Since $T(\tau)$ diverges as $\tau \rightarrow \infty$, there are no positive eigenvalues.

Determination of the Zero Eigenvalue: $\lambda = 0$

Assuming λ is zero, the differential equation for T becomes

$$\frac{T'}{T} = 0.$$

Multiply both sides by T.

$$T' = 0$$

The general solution is a constant.

$$T(\tau) = C_4$$

Now the differential equation for X will be solved.

$$\frac{1}{\xi X}\frac{d}{d\xi}\left(\xi X'\right) = 0$$

Multiply both sides by ξX.

$$\frac{d}{d\xi}\left(\xi X'\right) = 0$$

Integrate both sides with respect to ξ.

$$\xi X' = C_5$$

Divide both sides by ξ.

$$X' = \frac{C_5}{\xi}$$

In order to satisfy $X'(0) = 0$, we require $C_5 = 0$.

$$X' = 0$$

Integrate both sides with respect to ξ once more.

$$X(\xi) = C_6$$

Apply the second boundary condition to determine C_6.

$$X(1) = C_6 = 0$$

We obtain the trivial solution, $X(\xi) = 0$. Thus, zero is not an eigenvalue.

Determination of Negative Eigenvalues: $\lambda = -\gamma^2$

Assuming λ is negative, the differential equation for T becomes

$$\frac{T'}{T} = -\gamma^2.$$

Multiply both sides by T.

$$T' = -\gamma^2 T$$

The general solution can be written in terms of the exponential function.

$$T(\tau) = C_7 e^{-\gamma^2 \tau}$$

Now the differential equation for X will be solved.

$$\frac{1}{\xi X} \frac{d}{d\xi} \left(\xi X' \right) = -\gamma^2$$

Multiply both sides by $\xi^2 X$.

$$\xi \frac{d}{d\xi} \left(\xi X' \right) = -\gamma^2 \xi^2 X$$

Expand the left side.

$$\xi^2 X'' + \xi X' = -\gamma^2 \xi^2 X$$

Bring all terms to the left side.

$$\xi^2 X'' + \xi X' + \gamma^2 \xi^2 X = 0$$

This is known as the parametric form of Bessel's equation of order zero (with parameter γ). Its solution is written in terms of zero-order Bessel functions of the first and second kind, $J_0(\gamma\xi)$ and $Y_0(\gamma\xi)$, respectively.

$$X(\xi) = C_8 J_0(\gamma\xi) + C_9 Y_0(\gamma\xi)$$

Take a derivative of this solution with respect to ξ.

$$\begin{aligned} X'(\xi) &= C_8\gamma J_0'(\gamma\xi) + C_9\gamma Y_0'(\gamma\xi) \\ &= -C_8\gamma J_1(\gamma\xi) - C_9\gamma Y_1(\gamma\xi) \end{aligned}$$

Now we apply the boundary conditions to determine C_8 and C_9.

$$\begin{aligned} X'(0) &= -C_8\gamma J_1(0) - C_9\gamma Y_1(0) = 0 \\ X(1) &= C_8 J_0(\gamma) + C_9 Y_0(\gamma) = 0 \end{aligned}$$

Note that $J_1(0) = 0$ and $Y_1(0) = -\infty$, so the first equation is satisfied by setting $C_9 = 0$. That makes the second equation $C_8 J_0(\gamma) = 0$. In order to avoid getting the trivial solution, we insist that $C_8 \neq 0$. Then we have

$$J_0(\gamma) = 0.$$

We conclude that $\gamma = \alpha_n$ $(n = 1, 2, \ldots)$, where α_n is the nth zero of J_0. The eigenfunctions corresponding to these eigenvalues are

$$X(\xi) = C_8 J_0(\gamma\xi) \qquad \to \qquad X_n(\xi) = J_0(\alpha_n\xi), \quad n = 1, 2, \ldots.$$

Also,

$$T(\tau) = C_7 e^{-\gamma^2\tau} \qquad \to \qquad T_n(\tau) = e^{-\alpha_n^2\tau}, \quad n = 1, 2, \ldots.$$

According to the principle of superposition, the solution to the PDE for $\phi_t(\xi, \tau)$ is a linear combination of all products $T_n(\tau)X_n(\xi)$ over all the eigenvalues.

$$\phi_t(\xi, \tau) = \sum_{n=1}^{\infty} A_n e^{-\alpha_n^2\tau} J_0(\alpha_n\xi)$$

Part (c)

As a result, we have

$$\begin{aligned} \phi(\xi, \tau) &= \phi_\infty(\xi) - \phi_t(\xi, \tau) \\ &= 1 - \xi^2 - \sum_{n=1}^{\infty} A_n e^{-\alpha_n^2\tau} J_0(\alpha_n\xi). \end{aligned}$$

All that's left to do is to find A_n. We will use the initial condition for this.

$$\phi(\xi, 0) = 1 - \xi^2 - \sum_{n=1}^{\infty} A_n J_0(\alpha_n\xi) = 0$$

Isolate the series.

$$\sum_{n=1}^{\infty} A_n J_0(\alpha_n\xi) = 1 - \xi^2$$

To solve for A_n, multiply both sides by $J_0(\alpha_m\xi)\xi$, where m is an integer.

$$\sum_{n=1}^{\infty} A_n J_0(\alpha_n\xi) J_0(\alpha_m\xi)\xi = (1 - \xi^2) J_0(\alpha_m\xi)\xi$$

Now integrate both sides with respect to ξ from 0 to 1.

$$\int_0^1 \sum_{n=1}^{\infty} A_n J_0(\alpha_n \xi) J_0(\alpha_m \xi) \xi \, d\xi = \int_0^1 (1-\xi^2) J_0(\alpha_m \xi) \xi \, d\xi$$

Bring the integral inside the sum.

$$\sum_{n=1}^{\infty} A_n \int_0^1 J_0(\alpha_n \xi) J_0(\alpha_m \xi) \xi \, d\xi = \int_0^1 (1-\xi^2) J_0(\alpha_m \xi) \xi \, d\xi$$

Because the Bessel functions are orthogonal with one another on $[0, 1]$ with respect to the weight ξ, the integral on the left side is zero for $n \neq m$. The $n = m$ term is all that remains in the infinite series.

$$A_n \int_0^1 J_0^2(\alpha_n \xi) \xi \, d\xi = \int_0^1 (1-\xi^2) J_0(\alpha_n \xi) \xi \, d\xi$$

The integral on the left side is known. To solve the integral on the right, make the substitution $s = \alpha_n \xi$.

$$\begin{aligned} A_n \cdot \frac{1}{2} J_1^2(\alpha_n) &= \int_0^{\alpha_n} \left[1 - \left(\frac{s}{\alpha_n}\right)^2\right] J_0(s) \frac{s}{\alpha_n} \left(\frac{ds}{\alpha_n}\right) \\ &= \frac{1}{\alpha_n^4} \int_0^{\alpha_n} (\alpha_n^2 - s^2) J_0(s) s \, ds \end{aligned}$$

J_1 is the first-order Bessel function of the first kind. Use integration by parts.

$$\begin{aligned} u &= \alpha_n^2 - s^2 & dv &= J_0(s) s \, ds \\ du &= -2s \, ds & v &= J_1(s) s \end{aligned}$$

Apply the formula $\int u \, dv = uv - \int v \, du$.

$$\begin{aligned} A_n \cdot \frac{1}{2} J_1^2(\alpha_n) &= \frac{1}{\alpha_n^4} \left[\underbrace{(\alpha_n^2 - s^2) s J_1(s) \Big|_0^{\alpha_n}}_{=0} - \int_0^{\alpha_n} (-2s) J_1(s) s \, ds \right] \\ &= \frac{2}{\alpha_n^4} \int_0^{\alpha_n} s^2 J_1(s) \, ds \\ &= \frac{2}{\alpha_n^4} \cdot s^2 J_2(s) \Big|_0^{\alpha_n} \\ &= \frac{2}{\alpha_n^4} \cdot \alpha_n^2 J_2(\alpha_n) \\ &= \frac{2 J_2(\alpha_n)}{\alpha_n^2} \end{aligned}$$

Solve the equation for A_n.

$$A_n = \frac{4 J_2(\alpha_n)}{\alpha_n^2 J_1^2(\alpha_n)}$$

Here we apply a known property of Bessel functions, $J_{p+1}(x) = \frac{2p}{x} J_p(x) - J_{p-1}(x)$, to write J_2 in terms of J_1 and J_0.

$$\begin{aligned} A_n &= \frac{4}{\alpha_n^2 J_1^2(\alpha_n)} \left[\frac{2}{\alpha_n} J_1(\alpha_n) - \underbrace{J_0(\alpha_n)}_{=0} \right] \\ &= \frac{8}{\alpha_n^3 J_1(\alpha_n)} \end{aligned}$$

The solution for the dimensionless velocity ϕ is then

$$\begin{aligned} \phi(\xi, \tau) &= 1 - \xi^2 - \sum_{n=1}^{\infty} A_n e^{-\alpha_n^2 \tau} J_0(\alpha_n \xi) \\ &= 1 - \xi^2 - \sum_{n=1}^{\infty} \frac{8}{\alpha_n^3 J_1(\alpha_n)} e^{-\alpha_n^2 \tau} J_0(\alpha_n \xi). \end{aligned}$$

Therefore,

$$\phi(\xi, \tau) = (1 - \xi^2) - 8 \sum_{n=1}^{\infty} \frac{J_0(\alpha_n \xi)}{\alpha_n^3 J_1(\alpha_n)} \exp(-\alpha_n^2 \tau).$$

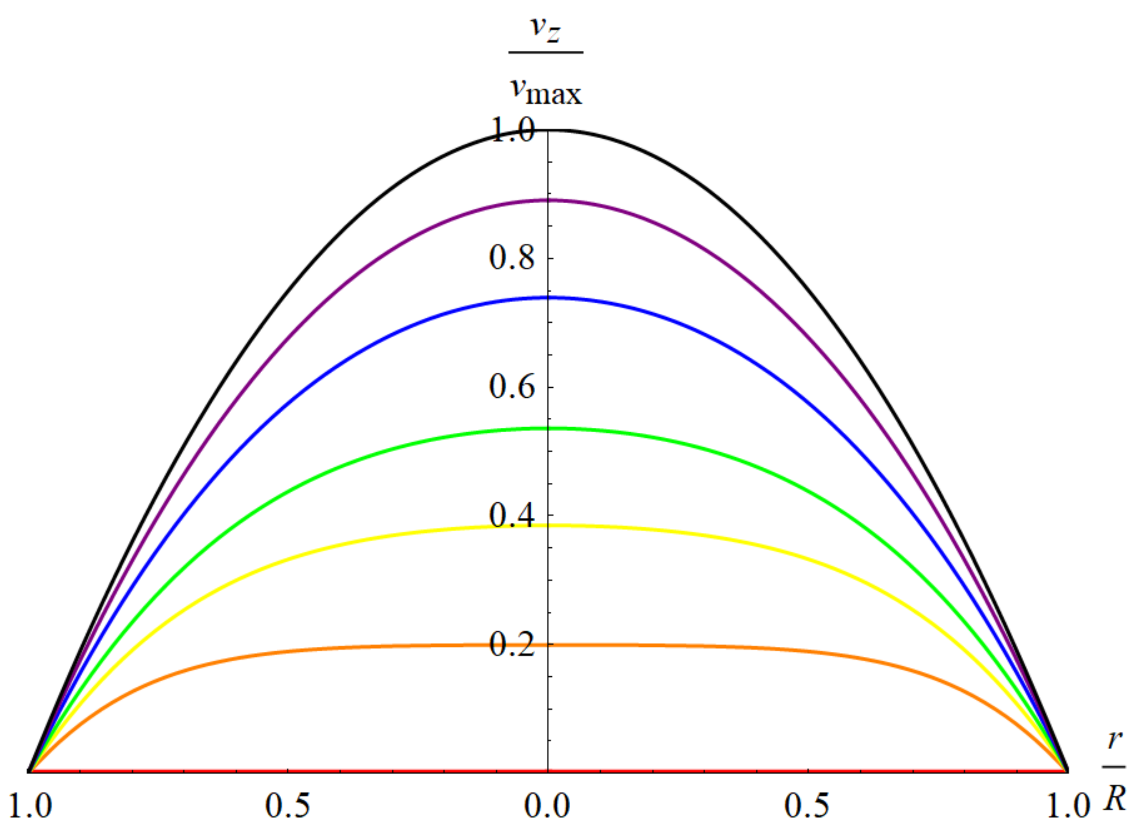

Figure 1: This is a plot of the dimensionless velocity $\phi = v_z/v_{\max}$ as a function of $\xi = r/R$ at various times. $\tau = 0$, $\tau = 0.05$, $\tau = 0.1$, $\tau = 0.15$, $\tau = 0.25$, $\tau = 0.4$, and $\tau = \infty$ correspond to the curves in red, orange, yellow, green, blue, purple, and black, respectively. These curves are only approximate, as only the first 10 terms in the infinite series have been used.