Problem 4D.5

Stream functions for three-dimensional flow.

(a) Show that the velocity functions $\rho\mathbf{v} = [\nabla \times \mathbf{A}]$ and $\rho\mathbf{v} = [(\nabla\psi_1) \times (\nabla\psi_2)]$ both satisfy the equation of continuity identically for steady flow. The second function also describes unsteady incompressible flows. The functions ψ_1, ψ_2, and $\mathbf{A}$ are arbitrary, except that their derivatives appearing in $(\nabla \cdot \rho\mathbf{v})$ must exist.

(b) Show that the expression $\mathbf{A}/\rho = -\boldsymbol{\delta}_3\psi/h_3$ reproduces the velocity components for the four incompressible flows of Table 4.2-1. Here h_3 and $\boldsymbol{\delta}_3$ are the scale factor and unit vector for the velocity component not shown in the table. (Read the general vector $\mathbf{v}$ of Eq. A.7-18 here as $\mathbf{A}$.)

(c) Show that the streamlines of $[(\nabla\psi_1) \times (\nabla\psi_2)]$ are given by the intersections of the surfaces ψ_1 = constant and ψ_2 = constant. Sketch such a pair of surfaces for the flow in Fig. 4.3-1.

(d) Use Stokes' theorem (Eq. A.5-4), and the definition of $\mathbf{A}$ from (a), to obtain an expression in terms of $\mathbf{A}$ for the mass flow rate through a surface S bounded by a closed curve C. Show that the vanishing of $\mathbf{v}$ on C does not imply the vanishing of $\mathbf{A}$ on C.

Solution

Part (a)

For steady flow, the continuity equation,

$$\frac{\partial\rho}{\partial t} = -\nabla \cdot \rho\mathbf{v},$$

reduces to

$$0 = -\nabla \cdot \rho\mathbf{v}.$$

The aim then is to show that the two provided functions satisfy

$$\nabla \cdot \rho\mathbf{v} = 0.$$

Begin with the first function $\rho\mathbf{v} = \nabla \times \mathbf{A}$.

$$\begin{aligned}
\nabla \cdot \rho\mathbf{v} &= \nabla \cdot (\nabla \times \mathbf{A}) \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left[\left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial}{\partial x_j}\right) \times \left(\sum_{k=1}^{3} \boldsymbol{\delta}_k A_k\right)\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left[\sum_{j=1}^{3}\sum_{k=1}^{3} (\boldsymbol{\delta}_j \times \boldsymbol{\delta}_k) \frac{\partial A_k}{\partial x_j}\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \cdot \left(\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \boldsymbol{\delta}_l \varepsilon_{jkl} \frac{\partial A_k}{\partial x_j}\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_l) \varepsilon_{jkl} \frac{\partial}{\partial x_i} \frac{\partial A_k}{\partial x_j}
\end{aligned}$$

Continue simplifying the right side.

$$\begin{aligned}
\nabla\cdot(\nabla\times\mathbf{A}) &= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\delta_{il}\varepsilon_{jkl}\frac{\partial}{\partial x_i}\frac{\partial A_k}{\partial x_j} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon_{jki}\frac{\partial}{\partial x_i}\frac{\partial A_k}{\partial x_j} \\
&= \sum_{j=1}^{3}\sum_{i=1}^{3}\sum_{k=1}^{3}\varepsilon_{ikj}\frac{\partial}{\partial x_j}\frac{\partial A_k}{\partial x_i} \qquad \text{(let } i \text{ be } j \text{ and let } j \text{ be } i\text{)} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon_{ikj}\frac{\partial}{\partial x_j}\frac{\partial A_k}{\partial x_i} \qquad \text{(constant limits means sums can be arranged however)} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}(-\varepsilon_{jki})\frac{\partial}{\partial x_j}\frac{\partial A_k}{\partial x_i} \\
&= -\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon_{jki}\frac{\partial}{\partial x_i}\frac{\partial A_k}{\partial x_j} \qquad \text{(use Clairaut's theorem to switch order of differentiation)} \\
&= 0
\end{aligned}$$

Now check that the second velocity function $\rho\mathbf{v} = (\nabla\psi_1)\times(\nabla\psi_2)$ satisfies the continuity equation.

$$\begin{aligned}
\nabla\cdot\rho\mathbf{v} &= \nabla\cdot[(\nabla\psi_1)\times(\nabla\psi_2)] \\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\cdot\left[\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial\psi_1}{\partial x_j}\right)\times\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k\frac{\partial\psi_2}{\partial x_k}\right)\right] \\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\cdot\left[\sum_{j=1}^{3}\sum_{k=1}^{3}(\boldsymbol{\delta}_j\times\boldsymbol{\delta}_k)\frac{\partial\psi_1}{\partial x_j}\frac{\partial\psi_2}{\partial x_k}\right] \\
&= \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\cdot\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_l\varepsilon_{jkl}\frac{\partial\psi_1}{\partial x_j}\frac{\partial\psi_2}{\partial x_k}\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_i\cdot\boldsymbol{\delta}_l)\varepsilon_{jkl}\frac{\partial}{\partial x_i}\left(\frac{\partial\psi_1}{\partial x_j}\frac{\partial\psi_2}{\partial x_k}\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\delta_{il}\varepsilon_{jkl}\left(\frac{\partial^2\psi_1}{\partial x_i\partial x_j}\frac{\partial\psi_2}{\partial x_k}+\frac{\partial\psi_1}{\partial x_j}\frac{\partial^2\psi_2}{\partial x_i\partial x_k}\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\delta_{il}\varepsilon_{jkl}\frac{\partial^2\psi_1}{\partial x_i\partial x_j}\frac{\partial\psi_2}{\partial x_k}+\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\delta_{il}\varepsilon_{jkl}\frac{\partial\psi_1}{\partial x_j}\frac{\partial^2\psi_2}{\partial x_i\partial x_k} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon_{jki}\frac{\partial^2\psi_1}{\partial x_i\partial x_j}\frac{\partial\psi_2}{\partial x_k}+\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon_{jki}\frac{\partial\psi_1}{\partial x_j}\frac{\partial^2\psi_2}{\partial x_i\partial x_k}
\end{aligned}$$

Let i be j and let j be i in the first triple sum, and let i be k and let k be i in the second triple sum.

$$\begin{aligned}
\nabla \cdot [(\nabla\psi_1) \times (\nabla\psi_2)] &= \sum_{j=1}^{3}\sum_{i=1}^{3}\sum_{k=1}^{3} \varepsilon_{ikj} \frac{\partial^2 \psi_1}{\partial x_j \partial x_i}\frac{\partial \psi_2}{\partial x_k} + \sum_{k=1}^{3}\sum_{j=1}^{3}\sum_{i=1}^{3} \varepsilon_{jik} \frac{\partial \psi_1}{\partial x_j}\frac{\partial^2 \psi_2}{\partial x_k \partial x_i} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ikj} \frac{\partial^2 \psi_1}{\partial x_j \partial x_i}\frac{\partial \psi_2}{\partial x_k} + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{jik} \frac{\partial \psi_1}{\partial x_j}\frac{\partial^2 \psi_2}{\partial x_k \partial x_i} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} (-\varepsilon_{jki}) \frac{\partial^2 \psi_1}{\partial x_j \partial x_i}\frac{\partial \psi_2}{\partial x_k} + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} (-\varepsilon_{jki}) \frac{\partial \psi_1}{\partial x_j}\frac{\partial^2 \psi_2}{\partial x_k \partial x_i} \\
&= -\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{jki} \frac{\partial^2 \psi_1}{\partial x_j \partial x_i}\frac{\partial \psi_2}{\partial x_k} + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{jki} \frac{\partial \psi_1}{\partial x_j}\frac{\partial^2 \psi_2}{\partial x_k \partial x_i}\right) \\
&= -\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{jki} \frac{\partial^2 \psi_1}{\partial x_i \partial x_j}\frac{\partial \psi_2}{\partial x_k} + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{jki} \frac{\partial \psi_1}{\partial x_j}\frac{\partial^2 \psi_2}{\partial x_i \partial x_k}\right) \\
&= 0
\end{aligned}$$

Part (b)

Substitute this given expression for $\mathbf{A}$ into the definition from part (a). Because the fluid is incompressible, ρ is constant and can be brought in front of the curl operator.

$$\begin{aligned}
\rho\mathbf{v} &= \nabla \times \mathbf{A} \\
&= \nabla \times \left(-\frac{\rho\psi}{h_3}\boldsymbol{\delta}_3\right) \\
&= -\rho\left(\nabla \times \frac{\psi}{h_3}\boldsymbol{\delta}_3\right)
\end{aligned}$$

Divide both sides by ρ.

$$\mathbf{v} = -\left(\nabla \times \frac{\psi}{h_3}\boldsymbol{\delta}_3\right)$$

Table 4.2-1 on page 123 shows how the stream function is related to the velocity for four different situations. In the first incompressible flow, the coordinate system is rectangular with no z-dependence, which means z is the third coordinate here: $\boldsymbol{\delta}_3 = \hat{\mathbf{z}}$. Looking at equation (F) on page 832, the scale factor for the z-component is $h_3 = 1$.

$$\mathbf{v} = -\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \psi \end{vmatrix} = -\left(\frac{\partial \psi}{\partial y}\hat{\mathbf{x}} - \frac{\partial \psi}{\partial x}\hat{\mathbf{y}}\right) = \left(-\frac{\partial \psi}{\partial y}\right)\hat{\mathbf{x}} + \left(\frac{\partial \psi}{\partial x}\right)\hat{\mathbf{y}}$$

Therefore,

$$\boxed{v_x = -\frac{\partial \psi}{\partial y} \quad \text{and} \quad v_y = \frac{\partial \psi}{\partial x}.}$$

In the second incompressible flow, the coordinate system is cylindrical with no z-dependence, which means z is the third coordinate here: $\boldsymbol{\delta}_3 = \hat{\mathbf{z}}$. Looking at equation (F) on page 834, the scale factor for the z-component is $h_3 = 1$. Expand the curl operator in cylindrical coordinates by using formulas (G), (H), and (I) on the same page.

$$\begin{aligned}
\mathbf{v} &= -\left(\nabla \times \frac{\psi}{h_3}\boldsymbol{\delta}_3\right) \\
&= -(\nabla \times \psi \hat{\mathbf{z}}) \\
&= -\left(\frac{1}{r}\frac{\partial \psi}{\partial \theta}\hat{\mathbf{r}} - \frac{\partial \psi}{\partial r}\hat{\boldsymbol{\theta}}\right) \\
&= \left(-\frac{1}{r}\frac{\partial \psi}{\partial \theta}\right)\hat{\mathbf{r}} + \left(\frac{\partial \psi}{\partial r}\right)\hat{\boldsymbol{\theta}}
\end{aligned}$$

Therefore,

$$\boxed{v_r = -\frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad \text{and} \quad v_\theta = \frac{\partial \psi}{\partial r}.}$$

In the third incompressible flow, the coordinate system is cylindrical with no θ-dependence, which means θ is the third coordinate here: $\boldsymbol{\delta}_3 = \hat{\boldsymbol{\theta}}$. Looking at equation (E) on page 834, the scale factor for the θ-component is $h_3 = r$. Expand the curl operator in cylindrical coordinates by using formulas (G), (H), and (I) on the same page.

$$\begin{aligned}
\mathbf{v} &= -\left(\nabla \times \frac{\psi}{h_3}\boldsymbol{\delta}_3\right) \\
&= -\left(\nabla \times \frac{\psi}{r}\hat{\boldsymbol{\theta}}\right) \\
&= -\left\{-\frac{\partial}{\partial z}\left(\frac{\psi}{r}\right)\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\psi}{r}\right)\right]\hat{\mathbf{z}}\right\} \\
&= -\left(-\frac{1}{r}\frac{\partial \psi}{\partial z}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial \psi}{\partial r}\hat{\mathbf{z}}\right) \\
&= \frac{1}{r}\frac{\partial \psi}{\partial z}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial \psi}{\partial r}\hat{\mathbf{z}}
\end{aligned}$$

Therefore,

$$\boxed{v_r = \frac{1}{r}\frac{\partial \psi}{\partial z} \quad \text{and} \quad v_z = -\frac{1}{r}\frac{\partial \psi}{\partial r}.}$$

In the fourth incompressible flow, the coordinate system is spherical with no ϕ-dependence, which means ϕ is the third coordinate here: $\boldsymbol{\delta}_3 = \hat{\boldsymbol{\phi}}$. Looking at equation (F) on page 836, the scale factor for the ϕ-component is $h_3 = r\sin\theta$. Expand the curl operator in spherical coordinates by using formulas (G), (H), and (I) on the same page.

$$\begin{aligned}
\mathbf{v} &= -\left(\nabla \times \frac{\psi}{h_3}\boldsymbol{\delta}_3\right) \\
&= -\left(\nabla \times \frac{\psi}{r\sin\theta}\hat{\boldsymbol{\phi}}\right) \\
&= -\left\{\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left[\left(\frac{\psi}{r\sin\theta}\right)\sin\theta\right]\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\psi}{r\sin\theta}\right)\right]\hat{\boldsymbol{\theta}}\right\} \\
&= -\left(\frac{1}{r^2\sin\theta}\frac{\partial \psi}{\partial \theta}\hat{\mathbf{r}} - \frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r}\hat{\boldsymbol{\theta}}\right)
\end{aligned}$$

Distribute the minus sign.

$$\mathbf{v} = -\frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta}\hat{\mathbf{r}} + \frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r}\hat{\boldsymbol{\theta}}$$

Therefore,

$$\boxed{v_r = -\frac{1}{r^2\sin\theta}\frac{\partial \psi}{\partial \theta} \quad \text{and} \quad v_\theta = \frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r}.}$$

Part (c)

Part (d)

The volumetric flow rate is given by the integral of the velocity field over the surface the fluid is flowing through.

$$\frac{dV}{dt} = \iint_S \mathbf{v}\cdot d\mathbf{S}$$

To get the mass flow rate through the surface, multiply both sides by the density ρ.

$$\rho\frac{dV}{dt} = \rho\iint_S \mathbf{v}\cdot d\mathbf{S}$$

ρ is constant, so it can be brought inside the operator on each side.

$$\frac{d(\rho V)}{dt} = \iint_S \rho\mathbf{v}\cdot d\mathbf{S}$$

Density times volume is the mass. Use $\rho\mathbf{v} = \nabla\times\mathbf{A}$ on the right side.

$$\frac{dm}{dt} = \iint_S (\nabla\times\mathbf{A})\cdot d\mathbf{S}$$

On the right side is the surface integral of a curl, so Stokes's theorem can be applied to change it to a closed loop integral over the surface's bounding curve C.

$$\boxed{\frac{dm}{dt} = \oint_C \mathbf{A}\cdot d\mathbf{s}}$$

If $\mathbf{v} = \mathbf{0}$ on C, then $\nabla\times\mathbf{A} = \mathbf{0}$ on C as well, which means that $\mathbf{A}$ can be written as the gradient of a scalar function f.

$$\mathbf{A} = \nabla f$$

f is an arbitrary function that is not necessarily zero. But the mass flow rate is zero on C by the fundamental theorem for line integrals because the beginning and end points are the same for a closed loop integral.

$$\begin{aligned}\frac{dm}{dt} &= \oint_C \nabla f\cdot d\mathbf{s}\\ &= f(a,b,c) - f(a,b,c)\\ &= 0\end{aligned}$$