Exercise 1

Unless the contrary is explicitly stated, all numbers that are mentioned in these exercises are understood to be real.

If r is rational ($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational.

Solution

Start by making the following assumptions.

$$r \text{ is rational.} \tag{A1}$$

$$x \text{ is irrational.} \tag{A2}$$

$$r \neq 0 \tag{A3}$$

Assumption (A1) implies that r can be written as the quotient of two integers, m and n, with $n \neq 0$.

$$r = \frac{m}{n}$$

By assumption (A3), $m \neq 0$.

The Sum

Consider the sum of r and x.

$$r + x = \frac{m}{n} + x = \frac{m}{n} + \frac{nx}{n} = \frac{m + nx}{n}$$

If $r + x$ were rational, then $m + nx$ would have to be an integer p.

$$m + nx = p$$

$$x = \frac{p - m}{n}$$

Since $p - m$ is an integer, this would mean x is rational, contradicting assumption (A2). Therefore, $r + x$ is irrational.

The Product

Consider the product of r and x.

$$rx = \frac{m}{n}x = \frac{mx}{n}$$

If rx were rational, then mx would have to be an integer q.

$$mx = q$$

$$x = \frac{q}{m}$$

But this would mean x is rational, contradicting assumption (A2). Therefore, rx is irrational.