Exercise 2

Unless the contrary is explicitly stated, all numbers that are mentioned in these exercises are understood to be real.

Prove that there is no rational number whose square is 12.

Solution

Start by showing there's no rational number whose square is 3. To do so, suppose that there is one.

$$r^2 = 3$$

Because r is rational, it's a ratio of two integers, m and $n \neq 0$, that has had all common factors cancelled.

$$\left(\frac{m}{n}\right)^2 = 3$$

$$\frac{m^2}{n^2} = 3$$

$$\frac{m^2}{3} = n^2$$

For n^2 to be an integer, m must have at least one factor of 3 in order to cancel the 3 in the denominator. Since m is squared, there is at least one factor of 3 left over in the numerator, meaning n also has at least one factor of 3. This contradicts the assumption that m and n have had all common factors cancelled. Therefore, there is no rational number whose square is 3.

Suppose now that there is a rational number k whose square is 12.

$$k^2 = 12$$

Because k is rational, it's a ratio of two integers, p and $q \neq 0$, that has had all common factors cancelled.

$$\left(\frac{p}{q}\right)^2 = 12$$

$$\frac{p^2}{q^2} = 12$$

$$p^2 = 12q^2 = 2(6q^2)$$

p^2 is even, so p is even: $p = 2s$, where s is another integer.

$$(2s)^2 = 12q^2$$

$$4s^2 = 12q^2$$

$$\frac{s^2}{q^2} = 3$$

$$\left(\frac{s}{q}\right)^2 = 3$$

This implies there is a rational number whose square is 3, which was shown to be false. Therefore, there is no rational number whose square is 12.