Problem 1.10

Use reduction of order to obtain the repeated root solution of $y''' - 3y'' + 3y' - y = 0$.

Solution

This ODE will be solved with reduction of order, also known as multiplicative substitution. This involves finding a solution by inspection, making a prescribed substitution, and solving the reduced, simplified ODE. Because the equation is third order, we expect there to be three linearly independent solutions and three arbitrary constants in the solution. Notice that the coefficients on the left side all add to 0. Thus, one solution to this ODE is $y_1(x) = e^x$. Let's verify this.

$$\begin{aligned} y_1''' - 3y_1'' + 3y_1' - y_1 &= e^x - 3e^x + 3e^x - e^x \\ &= 0 \end{aligned}$$

The next step is to make the prescribed substitution, $y = y_1 u(x)$, into the ODE. Find expressions for y' and y'' and y''' in terms of the new variable u.

$$\begin{aligned} y &= e^x u(x) \\ y' &= e^x u + e^x u' = e^x(u' + u) \\ y'' &= e^x(u' + u) + e^x(u'' + u') = e^x(u'' + 2u' + u) \\ y''' &= e^x(u'' + 2u' + u) + e^x(u''' + 2u'' + u') = e^x(u''' + 3u'' + 3u' + u) \end{aligned}$$

Now plug these into the ODE.

$$e^x(u''' + 3u'' + 3u' + u) - 3e^x(u'' + 2u' + u) + 3e^x(u' + u) - e^x u = 0$$

Divide both sides by e^x and combine like-terms.

$$u''' = 0$$

This equation can be solved by simply integrating both sides with respect to x three times.

$$u'' = C_1$$

$$u' = C_1 x + C_2$$

$$u(x) = \frac{1}{2}C_1 x^2 + C_2 x + C_3$$

To eliminate the constant, 1/2, introduce new constants of integration, A and B and C.

$$u(x) = Ax^2 + Bx + C$$

Now that we know $u(x)$, we can solve for $y(x)$, the solution we care about.

$$y(x) = e^x u(x)$$

Therefore,

$$y(x) = e^x(Ax^2 + Bx + C).$$

Note that the first solution we found by inspection, e^x, is included in the answer for $y(x)$.