Problem 1.24

Show that a necessary and sufficient condition that $M(x,y) + N(x,y)\,dy/dx = 0$ be exact is $\partial M/\partial y = \partial N/\partial x$.

Solution

What we have to show is that $M(x,y) + N(x,y)\,dy/dx = 0$ is exact implies $\partial M/\partial y = \partial N/\partial x$ and vice-versa. That is, there are two parts to this proof:

$$\text{Part I}: \quad M(x,y) + N(x,y)\frac{dy}{dx} = 0 \text{ is exact} \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\text{Part II}: \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \Rightarrow \quad M(x,y) + N(x,y)\frac{dy}{dx} = 0 \text{ is exact.}$$

Part I

The fact that $M(x,y) + N(x,y)\,dy/dx = 0$ is exact means there exists a potential function $\phi = \phi(x,y)$ such that

$$\frac{\partial \phi}{\partial x} = M(x,y) \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N(x,y). \tag{2}$$

Differentiate both sides of equation (1) partially with respect to y and differentiate both sides of equation (2) partially with respect to x.

$$\frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial M}{\partial y}$$

$$\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial N}{\partial x}.$$

According to Clairaut's theorem,

$$\frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial^2 \phi}{\partial x \partial y},$$

provided that these second derivatives are continuous. Therefore,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Part II

Starting with the premise,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \tag{3}$$

our aim here is to show that there exists a potential function $\phi = \phi(x,y)$ such that

$$\frac{\partial \phi}{\partial x} = M(x,y) \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N(x,y). \tag{2}$$

We will use these two equations to find this function. Integrate both sides of equation (1) partially with respect to x.

$$\phi(x,y) = \int^x M(s,y)\,ds + f(y), \tag{4}$$

where f is an arbitrary function of y. In order to determine f, differentiate both sides of this result partially with respect to y.

$$\frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}\int^x M(s,y)\,ds + \frac{df}{dy}$$

According to equation (2), we have

$$N(x,y) = \frac{\partial}{\partial y}\int^x M(s,y)\,ds + \frac{df}{dy}.$$

Solve this equation for df/dy.

$$\frac{df}{dy} = N(x,y) - \frac{\partial}{\partial y}\int^x M(s,y)\,ds \tag{5}$$

We can solve this for f by integrating both sides with respect to y. Before we do that, though, we have to show that the right side is a function of y only, as we have a total derivative with respect to y on the left side. This can be done by differentiating the right side partially with respect to x and showing that it is equal to 0.

$$\begin{aligned}
\frac{\partial}{\partial x}\left[N(x,y) - \frac{\partial}{\partial y}\int^x M(s,y)\,ds\right] &= \frac{\partial N}{\partial x} - \frac{\partial^2}{\partial x \partial y}\int^x M(s,y)\,ds \\
&= \frac{\partial N}{\partial x} - \frac{\partial^2}{\partial y \partial x}\int^x M(s,y)\,ds \\
&= \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\frac{\partial}{\partial x}\int^x M(s,y)\,ds \\
&= \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}M(x,y) \\
&= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \\
&= \frac{\partial N}{\partial x} - \frac{\partial N}{\partial x} \\
&= 0,
\end{aligned}$$

where equation (3) was used to change $\partial M/\partial y$ to $\partial N/\partial x$. Now we can integrate both sides of equation (5) with respect to y.

$$f(y) = \int^y N(x,r)\,dr + \int^y \frac{\partial}{\partial r}\int^x M(s,r)\,ds\,dr + C,$$

where C is an arbitrary constant. Plugging this result into equation (4), we thus have a potential function that satisfies equations (1) and (2), which means $M(x,y) + N(x,y)\,dy/dx = 0$ is exact.

$$\phi(x,y) = \int^x M(s,y)\,ds + \int^y N(x,r)\,dr + \int^y \frac{\partial}{\partial r}\int^x M(s,r)\,ds\,dr + C$$

The two parts of the proof are complete. Therefore, a necessary and sufficient condition that $M(x,y) + N(x,y)\,dy/dx = 0$ be exact is $\partial M/\partial y = \partial N/\partial x$.