

## Problem 1.28

Discuss the existence and uniqueness of solutions to the initial-value problem  $y' = \sqrt{1 - y^2}$  [ $y(0) = a$ ], for all initial values  $a$ . Is there a unique solution if  $a = 1$ ?

### Solution

The square root function on the right side exists and is continuous for any neighborhood of  $x$  and  $y$  when  $-1 \leq y \leq 1$ . However, the derivative of it with respect to  $y$ ,

$$\frac{\partial}{\partial y} \sqrt{1 - y^2} = \frac{1}{2}(1 - y^2)^{-1/2}(-2y) = -\frac{y}{\sqrt{1 - y^2}},$$

is not continuous when  $y$  is in the neighborhood of 1 or  $-1$ . The conditions of the existence and uniqueness theorem are not satisfied here, so uniqueness is not guaranteed. Consequently, for the initial-value problem,  $y' = \sqrt{1 - y^2}$  [ $y(0) = 1$ ], there may be more than one solution. Indeed, by inspection we see that  $y(x) = 1$  satisfies it and by separation of variables a second solution can be obtained.

$$\frac{dy}{\sqrt{1 - y^2}} = dx$$

Integrate both sides.

$$\arcsin y = x + C$$

Take the sine of both sides.

$$y(x) = \sin(x + C)$$

Determine the constant of integration by using the initial condition.

$$y(0) = \sin(C) = 1 \quad \rightarrow \quad C = \frac{\pi}{2}$$

Hence,

$$y(x) = \sin\left(x + \frac{\pi}{2}\right)$$

is a second solution. The discussion is summarized below.

$$\begin{cases} a^2 > 1 & \text{No solution to the IVP exists} \\ a^2 = 1 & \text{A solution to the IVP exists but is not unique} \\ a^2 < 1 & \text{A solution to the IVP exists and is unique} \end{cases}$$