Problem 1.31

Solve the following differential equations:

(*a*) $y' = y/x + 1/y$;

Solution

Multiply both sides of the ODE by y.

$$yy' = \frac{y^2}{x} + 1$$

Rewrite the left side as follows.

$$\frac{d}{dx}\left(\frac{1}{2}y^2\right) = \frac{y^2}{x} + 1$$

Bring the constant out of the derivative and move the y^2 term to the left.

$$\frac{1}{2}\frac{d}{dx}\left(y^2\right) - \frac{y^2}{x} = 1$$

Multiply both sides by 2 to get rid of the 1/2 factor.

$$\frac{d}{dx}(y^2) - \frac{2}{x}y^2 = 2$$

This is a first-order inhomogeneous ODE for y^2 that can be solved with an integrating factor I.

$$I = e^{\int^x -\frac{2}{s}\,ds} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2}$$

Multiply both sides of the equation by the integrating factor.

$$\frac{1}{x^2}\frac{d}{dx}(y^2) - \frac{2}{x^3}y^2 = \frac{2}{x^2}$$

The left side is now exact and can be written as $d/dx(Iy^2)$ as a result of the product rule.

$$\frac{d}{dx}\left(\frac{1}{x^2}y^2\right) = \frac{2}{x^2}$$

Integrate both sides with respect to x.

$$\frac{1}{x^2}y^2 = -\frac{2}{x} + C,$$

where C is an arbitrary constant. Multiply both sides by x^2.

$$y^2 = -2x + Cx^2$$

Therefore,

$$y(x) = \pm\sqrt{-2x + Cx^2}.$$