Problem 1.31

Solve the following differential equations:

(g) $xy' = y(1 - \ln x + \ln y)$;

Solution

Divide both sides by x and combine the logarithms on the right side.

$$y' = \frac{y}{x}\left(1 - \ln\frac{y}{x}\right)$$

The right side suggests the subsitution,

$$u = \frac{y}{x} \quad \rightarrow \quad xu = y$$

$$u + x\frac{du}{dx} = \frac{dy}{dx}.$$

Plug these expressions into the ODE.

$$u + x\frac{du}{dx} = u(1 - \ln u)$$

Subtract u from both sides.

$$x\frac{du}{dx} = -u\ln u$$

This ODE can be solved by separation of variables.

$$\frac{du}{u\ln u} = -\frac{dx}{x}$$

Integrate both sides.

$$\int\frac{du}{u\ln u} = -\ln|x| + C$$

Use the following substitution to evaluate the integral on the left.

$$v = \ln u$$

$$dv = \frac{du}{u}$$

The integral becomes

$$\int\frac{dv}{v} = -\ln|x| + C.$$

So we have

$$\ln|v| = -\ln|x| + C.$$

Exponentiate both sides.

$$|v| = |x|^{-1}e^C$$

Introduce $\pm$ on the right side to eliminate the absolute value sign on the left.

$$v = \frac{\pm e^C}{|x|}$$

Use a new arbitrary constant A.

$$v = \frac{A}{|x|}$$

It's because A is arbitrary that we can drop the absolute value sign in the denominator. Change back to the variable u.

$$\ln u = \frac{A}{x}$$

Exponentiate both sides.

$$u = e^{A/x}$$

Now change back to the original variable y.

$$\frac{y}{x} = e^{A/x}$$

Multiply both sides by x to solve for y. Therefore,

$$y(x) = xe^{A/x}.$$